Chapter 4: Production: Making Things to Sell

I. Introduction: Household Production versus Production for Sale

When economists talk about production, they refer to activities undertaken to transform a group of materials and services into others of greater value. In a commercial society, the individuals or organizations that produce a good or services plan to sell it (or them) to other persons, groups, or organizations. Production is not just adding value or providing services but making stuff and providing services that others in in their communities or trading networks are willing (and able) to pay for. Of course, not everything that is produced is for sale, nor is everything sold in markets newly produced for sale.

Preexisting things may be sold. Land, buildings, classic artwork , used cars, used books, and used cell phones, for example, are routinely resold. So, there are markets for previously used products as well as those newly produced for sale. .However, most of the products sold in contemporary markets are newly produced. The supply functions of the goods analyzed in chapter 3 were all for goods that were newly produced. They were produced because the value added through the productive efforts of firms increase their value sufficiently that profits could be realized by selling them. If sufficient value was not added to the inputs used, the cost of the inputs would exceed the market value of the products produced, and losses would be realized instead of profits by selling them. Such firms and production processes would not long survive.

This is not to say that the term "production" is never used to describe the manner in which goods that are not sold are brought into existence. For example, that term is also used by economists to describe the activities of persons who raise some their own vegetables in gardens, bake bread, or construct sheds to store tools in their backyards, without any intent to sell them. Such activities are often termed **household production** (Becker, 1965).

In the periods and places before commercial societies emerged, a good deal, perhaps most, of a family's time, knowledge, and materials were devoted to such productive activities. Foodstuffs were homegrown or the product of hunting and gathering in nearby forests. Thread was home spun, and cloth created on looms at home or by nearby weavers. A good deal of clothing was homemade. Cooking was conducted over fires fueled by wood harvested from nearby woodlands, and so forth.

Although a lot of time was spent producing goods and services in the period before commerce emerged, relatively little of it was produced for sale.

Thus, relatively little of household production is relevant for price theory. Nonetheless, it can be modeled using rational-choice models. Household production is a utility maximizing choice (an allocation of time), rather than a profit maximizing one. Moreover, household production remains an important enterprise today, even if it is less so than in past centuries. Most of what we bring home from a grocery and building stores is used in household production—as the "inputs" of bread, peanut butter, and jelly, together with labor, and capital in the form of knives, plates, and napkins are used to construct a peanut butter and jelly sandwich. Additional equipment and more labor are used to clean up afterwards. Lumber purchased by consumers may be transformed into bookshelves or tables using one's own labor and capital (saws, measuring tapes, hammers, clamps, and so on).

In cases in which household production takes place in a commercial society, household production indirectly affects prices through demands for inputs and effects on the markets for final products. Conversely, market prices affect decisions to engage in household production. Household production is undertaken because the quality of the final product produced at home is deemed superior to that which can be purchased in markets for the same cost in time and money.

The main difference between household production and production for sale in markets is a difference in objectives. In household production, the aim is to increase utility directly either because the production process itself is valued, or indirectly because the outputs are regarded to be more pleasing or less expensive than substitutes that could have been purchased in markets. The scope of household commerce tends to diminish when it is easier to purchase goods by trading labor for money and then money for the desired goods and services than it is to produce them at home. Indeed, many products that are readily available in market cannot be produced at home.

Production for sale takes place when the net income from producing and selling goods yields more utility than engaging in household production. This is normally because specialization and team production decrease the cost of producing goods and services relative to household production.

Intermediate cases also exist in which a person or group of persons produce some things for themselves (as with vegetable gardens, hunters, or carpenters) and sell part of the things produced to their acquaintances or in local markets. Such intermediate cases were likely to be the first commercial producers. As markets extended, such intermediate cases gradually became fully commercial—devoting most of their resources for production for sale and less for household production. That, for example, accounts for the transition from subsistence farming to contemporary commercial farms. Larger commercial organizations became commonplace in the nineteenth century during the period in which dense and ubiquitous commercial networks emerged. Very luge commercial organizations are relatively new—less than a couple of centuries old. Yet, a place remains for the older smaller types of economic organizations. Family-based firms are still commonplace and account for roughly half of the jobs in today's commercial societies.

It is the market relationships of late nineteenth and early twentieth century commercial societies in the West that neoclassical economics emerged to explain. It was largely during the nineteenth century that production within both relatively small and large organizations devoted to selling their output(s) became commonplace and household production for own use diminished in importance. More and more persons "hired themselves out for wages" and used the proceeds to purchase most of the necessities of life as well as other goods consumed for pleasure.

Together, the neoclassical economic theory of supply and production characterize the processes through which new things or services are created and sold in markets. They do so in a somewhat abstract and general fashion. They do not focus on the details of marketing and production, but rather common features of those activities. The neoclassical theory of production models the choices about production that generate a firm's cost functions. Their cost functions, in turn, largely determine a firm's output decisions. Only profitable outputs are intentionally produced. And to the extent that firms successfully maximize their profits, only least-cost methods of production are used.

II. The Simplest Integrated Model of Production and Supply

The simplest methods of production use only a single variable input such as labor. Other inputs may be fixed in some way, either by nature, as with gathering fruit from wildly growing fruit trees, or because the other factors are generally held constant as production increases, such as might be said of an ax used to cut some trees down to build a fire, a house, or clear a pasture. One may need an ax to produce firewood, but after an individual has an ax, the rest is all labor.

The production function in such cases can be modeled with a one-dimensional function such as Q = q(L) where function q is assumed to be strictly concave—e.g., to exhibit diminishing returns over the full or relevant domain of production. Total variable costs would simply be C=wL, w is the wage rate (or opportunity cost) of labor Production costs in a commercial society include all expenditures on inputs. (For household production, the cost of using labor in a particular activity is its opportunity cost in other uses.)

The cost functions that we used in chapter 3 were all functions of the firm's output, a specification that was important for the firm's decision about how much output to produce and bring to market. To create a cost function in terms of output rather than inputs, we need to determine the relationship between output and inputs. If, for example, $Q = aL^b$, with b<1, then the inverse function—the labor required to produce a given output, is the solution for L as a function of Q, which a bit of algebra finds is: $L = (Q/a)^{1/b}$.

Given that relationship we can write the cost function as a function of output by substituting the inverse function that describes how much labor is being used as a function of quantity into the firm's cost function. In the first case, $C = wL = w\ (Q/a)^{1/b}$. Algebraically, this equation looks a bit complicated, but notice that it is very similar to the simplest of the exponential cost functions that we used in Chapter 3 (e.g. $C=aQ^b$)

The firm's profit maximizing output depends in part on the type of choice setting faced by the firm. When it is a price taker, the prevailing market price is the marginal revenue generated from selling additional its output. In this case, the profit function is simply, $\Pi = PQ - w(Q/a)^{1/b}$.

To characterize the firm's optimal output, differentiate the profit function with respect to Q, and set the result equal to zero. That relationship characterizes, Q*, the output that maximize profit. In the simple case being worked through, this yields $P - w\ (1/b)(Q/a)^{(1/b)-1}. = O$

The firms supply function is found by solving that relationship for Q*, which requires a bit of algebra. $P = w\ (1/b)(Q/a)^{(1/b)-1} \rightarrow bP/w = (Q/a)^{(1-b)/b} \rightarrow (bP/w)^{b/(1-b)} = (Q/a)$ which implies that $Q^* = (1/a)(bP/w)^{b/(1-b)}$. This is the firm's supply function given the production function assumed. Note that supply rises with price and falls as the input price increases.

Deriving the supply function in the general case of production with a single input follows the same steps, although with less algebra. If the production function is Q=q(L), then the inverse of the production function is characterized using the implicit function theorem on 0=Q-q(L). That function is written as $L=q^{-1}(Q)$, where the (-1) denotes an inverse in this case rather than an exponent of the q function. The firm's cost function is simply $C= w\, q^{-1}(Q)$ and its profit function (if a price taker) is simply $\Pi = PQ - w\, q^{-1}(Q)$.

Notice that this is identical with the model of the firm's choice used to generate the supply function in chapter 3, except the cost function is now explicitly derived from the firm's production process—rather than simply assumed—and the cost function now explicitly takes account of the input price. To characterize the firm's supply curve we, again, differentiation its profit function with respect to Q and set the result equal to zero. $d\Pi/dQ= P - w(dq^{-1}/dQ) = 0 \equiv H$. The first term (P) is the firm's marginal revenue and the second $[w(dq^{-1}/dQ)]$ is its marginal cost.

To characterize the firm's supply function, we appeal to the implicit function theorem, which implies that Q* (the quantity that satisfies the first order condition) can be written as: $Q^* = s(P, w)$. This is the firm's supply function.

The implicit function differentiation rule can be used to find the effects of price and wage rates on the firm's output. The first order condition is used as the relevant zero function.

$$dQ^*/dP = (dH/dP) \,/\, (- dH/dQ^*) = 1/-(d^2\Pi/dQ^2) > 0$$

$$dQ^*/dw = (dH/dw) \,/\, (- dH/dQ^*) = -(dq^{-1}/dQ)/-(d^2\Pi/dQ^2) < 0$$

The denominator has been written in terms of derivatives of the profit function to simplify and shorten the derivation. The second derivative of the profit function is negative if it is strictly concave, so the denominators of both of these derivatives are positive (because of the leading negative sign).

The numerator of the derivative with respect to price is simply 1, which is positive. So, the overall effect of an increase in price on the quantity supplied is positive, as in the concrete functional form case. The numerator of the derivative with respect to input price w is the negative of the slope of the inverse production function. The inverse production function and production function have the same slope, which is positive in this case because marginal product has been assumed to be greater than zero over the range of interest. Thus, the overall effect of an increase in wage rates on the quantity supplied is negative. An increase in wage rates shifts the supply function (curve) back to the left.. These are quite general relationships that follow whenever the production function is subject to diminishing returns overall or at least within the domain of interest.

In terms of the mathematics, most of the complexity is generated by the necessity of generating a cost function that specifies a firm's total cost as a function of output levels. This method of deriving a firm's supply function is most similar to that used in chapter 3 to characterize a firm's supply decision.

Note that we could have reversed the order of the math in cases where production involves just one input. Profit could have been written in terms of labor usage as $\Pi = Pq(L) - wL$. The ideal level of labor is that which maximizes profit, which can be determined by differentiating the profit function with respect to L and setting the result equal to zero. This yields: $P\,(dQ/dL) - w = 0$. The first order condition implies that the firm will hire labor up to the point where its marginal revenue product, $P\,(dQ/dL)$ equals the wage rate (w). L* can be characterized using the implicit function theorem as $L^* = g(P, w)$. This is **the firm's demand for labor** which varies with its price (w) and the selling price of the output (P). The output associated with that choice can be characterized using the production function, as with $Q^* = q(L^*)$. Given the variables in L*, the implicit function theorem implies that Q* can be written as $Q^* = s(P, w)$ as before.

Unfortunately, the very clean and general derivation can only be undertaken for one input production functions. Nonetheless, the result that firms hire inputs up to the point where their marginal product times the price of the output equals the price of the input, is quite general and a very useful rule of thumb to keep in one's mind. The longer somewhat more complex calculations undertaken first are required the firm uses a multiple input production function. These are the cases focused on in the rest of the chapter.

III. The Geometry of Production Requiring More than One Input

The geometry used to model production choices is similar to that used to characterize a consumer's choices among several goods and services. However, instead of attempting to maximize utility given a budget constraint, firms attempt to maximize production for given levels of expenditures on inputs. One difference between consumer and firm choices is the assumption that a firm's expenditures are in a sense unlimited because of access to capital markets and so it can choose the expenditure level to undertake. That will normally be the one that produces the output that maximizes profits as characterized in chapter 3. However, to determine the profit maximizing output level, first the firm's cost function has to be determined, which in turn, requires the method of production to be determined.

Figure 4.1 illustrates a firm's cost minimizing use of inputs for three possible outputs. The C-shaped curves are referred to as isoquants, each curve represents the various combinations of inputs that can be used to produce a particular output (an iso (single) quantity (quant) of output). The two inputs are usually assumed to be capital (equipment) and labor, although others may be more relevant for the product of interest.

As in geometric representations of consumer choices using utility functions, graphical representations are limited to production processes with two inputs because paper and computer screens are two dimensional. The straight lines (iso-cost curves) resemble budget constraints because each iso-cost line represents a particular total expenditure on two goods (L and K) at given prices (w and r). In this case, they are various combinations of inputs that can be used to produce a product or service for sale in markets. Each iso-cost line characterizes the possible combinations of inputs that could be purchased with a given expenditure level (total cost).

Firms naturally attempt to minimize the cost of producing each possible output level. This occurs at **tangency points between the isocost and isoquant lines,** three of which are illustrated. There is a total cost (total expenditure on inputs) and an output level at each point of tangency. Together the tangencies associated with different outputs, characterize the cost (minimum cost) of producing those outputs—which is to say they characterize the firm's **total cost function**. The production process adopted is the one that which minimizes the cost of the profit-maximizing output for the firm, given input prices and the selling price of the final good. The diagram that it costs amount C_1 to produce quantity Q_1, amount C_2 to produce quantity Q_2, and amount C_3 to produce quantity Q_3. Thus (C_1, Q_1), (C_2,Q_2) and (C_3.Q_3) are all points on the firm's cost function.

That output of the firm is not determined by the diagram drawn, but the cost function characterized by the diagram. The cost function is used by the firm to determine its profit maximizing output, Q*. The production method used is the one on the Q* isoquant that is tangent to the iso-cost line. It is the least cost method of producing Q* units of the good. The method of production is indicated by the combination of inputs that the firm uses.

Figure 4.1 Relationship between Output, Cost and Input Mix for a Price-Taking Firm

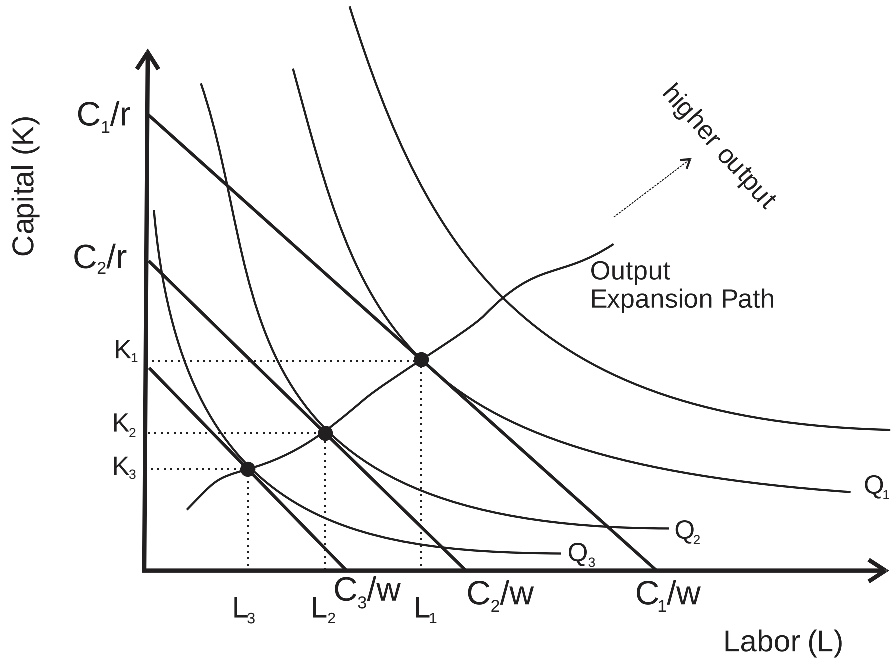

Figure 4.1 thus shows in principle how firms produce their products when they use a particular technology or combination of technologies is used for production. No innovation is being represented.

Price taking firms face particular input prices (w, wages, and r, the rental cost of capital) and a particular price for its output (P). If any of these parameters of a firm's choice setting change, the optimal combinations of inputs used to produce its output will also change, and, thus, so will its cost curve and cost function.

For example, with the assumed C-shaped isoquants, firms will use more capital and less labor when the price of labor increases, which causes their total and marginal cost for production will tend to increase. The latter occurs because as w increases the slopes of the iso-cost lines change and the least cost method of producing a particular output such as Q* tends to increase. If we knew the numbers associated with the iso cost curves and isoquants, we could have precisely characterized total cost and marginal cost. This would require knowing both the precise production function used and input prices.

Nonetheless, the qualitative results generated by diagrams similar to figure 4.1, provide **a number of insights**. Profit maximizing firms produce with input mixes that equate **marginal rates**

of technical substitution (the slope of an isoquant) with their relative prices (the slope of an iso cost line). As output expands, the mix of inputs employed may change. More capital goods, for example, might be employed. On the other hand, the geometry most often used in such diagrams suggests that input use tends to be roughly proportional to output—although exceptions to that rule of thumb can also be drawn.

For example, the adoption of more capital-intensive methods as output expands requires somewhat counter intuitive shapes and locations for the isoquants—but ones that can be drawn. It turned out that in the nineteenth century, such curves were associated with the production of a number of products—as with the production of steel, many chemicals, and electricity. Thus, special cases of this geometry are often relevant for real-world businesses. Older small businesses, tended to have more linear output expansion paths, where all inputs increased roughly in proportion to output.

The **distinction between the long run and short run adjustments** can also be explained by the same model. If one (or more) of the factors of production is fixed in the short run, the cost of increasing output tends to be higher than when it is not. Normally the factor that is most time-consuming to adjust in the short run is capital. Erecting new factories and equipping then using takes longer than adding a few more people to the production process. Although it should be noted that some types of labor may also take a long time to change—as with some types of firm specific human capital and others that require long training periods, such as doctors, lawyers, and economists. In the usual case, this means that short run supply decisions cannot use capital-labor substitution to economize on rising labor costs in the short run, although they can do so in the long run. The second less common problem of fixed labor can be represented in a similar way. Both problems for firms imply that reaching a higher isoquant will require greater expenditures on inputs in the short than in the long run. Thus, short run marginal cost curves tend to steeper than long run marginal cost firms.

Diagrams of this sort can also be used to characterize production functions that are actually quite common, but that do not lend themselves to the tools of calculus. Namely production processes in which discontinuities or **impossibilities of substitution** occur (**Leontief production functions**). For example, production may require that particular ratios of inputs be used to produce the desired output. Bicycles, for example, always have two wheels (here partly by definition, but also partly because of the physics of riding them). If the price of wheels or hubs changes, no substitution (holding quality constant) of, say, pedals or seats for wheels is possible. Each bicycle needs two wheels, two hubs, and two tires. The isoquants for such products tend to be L-shaped rather than C-shaped and production takes place at the kink (where the vertical of the L intersects with the horizontal part). All others possible uses of inputs are wasteful and would be avoided since they do not increase the output of the final good (bicycles). The discontinuity in the slope of the isoquant at the "corner" of the L limits the extent to which the optimization methods of calculus can be employed to account for such production.

Nonetheless, the calculus approach shed useful light on aspects of production where substitution among inputs and associated methods of production are possible.

IV. The Calculus of Production Using Concrete Production Functions

Just as the geometric representation of a firm's production decisions is similar to that used for modeling consumer choices. The calculus-based model of a firm's production decision is very similar to that of consumer theory. Even the family of functions focused on tend to be similar. Cobb-Douglas and multiplicative exponential functions are among the most common families of functions used. Unfortunately, the results are somewhat more difficult to derive and more complex for production than they are for consumption. This occurs mostly because there are two stages to the choices modeled. First the production stage, which is used to characterize a cost function. And second, a supply stage using that cost function, more or less as done in chapter 3.

As in Figure 4.1, we'll begin by analyzing a two-input production process, where the inputs are labor and capital. The cost of the products or services produced are determined by market prices for inputs and the use rates of those inputs: $C = wL^* + rK^*$, where L^* and K^* are the usage of labor and capital to produce the profit maximizing output in the least cost way possible under the existing technology... As in the geometric case, the cost depends on factor prices (w and r) and the production function used to produce the products to be sold.

Suppose also that the firm's output from the use of labor and capital is $Q = L^eK^f$, where both e and f are greater than 0, but less than 1. The "technology" of production is represented both by the inputs required and the exponents of the production function assumed. The production process is regarded to be exogenous (beyond the control of the firm) and constant during the period of interest (sometimes called the planning horizon. The prices of inputs (w and r) are also parameters of the choice setting when firms are price takers in input markets.

The assumption that the sum of the exponents is less than 1 and greater than zero implies that each factor of production contributes to output but also exhibits diminishing marginal returns. The overall production process also exhibits diminishing marginal returns. In such cases, the associated marginal cost function is upward sloping—as developed below. If their sum is exactly 1, this would be a **Cobb-Douglas** production function, and the overall production process would exhibit constant returns to scale.

The firm's cost function can be derived either by minimizing the cost of a given output or by maximizing the output achieved for a given cost. These approaches can be regarded as the "**dual**" of the other. The results should be identical to one another, and so the best approach is the one that appears most likely to be error free.

We'll first use the maximizing output for a given cost approach, because this is similar to that used for deriving consumer demand and is, in some sense, the "natural" way to characterize a firm's cost function. Perhaps surprisingly, the results are more difficult to derive in this way and more difficult to interpret than the less intuitive solution to the dual problem—as we will see. Both solutions obtained are derived through a process similar to consumer choice problem analyzed in chapter 2, and they both characterize a firm's demand for inputs for a given overall expenditure on inputs, although the objective function that we are modelling are with respect to profits rather than utility.

Maximizing Output for a Given Cost

The Lagrange approach is a bit easier than the substitution method in this case, because the assumed production function is a multiplicative-exponential function, and the direct cost considerations are linear. However, as usual, the individual equations are more difficult to interpret than those obtained using the substitution method. Since L is being used to characterize the quantity of labor used in production, we'll use a "script L" for the symbol representing the Lagrange function ($\mathscr{L}$).

The Lagrangian equation associated with maximizing the output achieved at a given cost is.

$$\mathscr{L} = L^e K^f + \lambda(C - wL + rK) \qquad (4.1)$$

As in the consumer choice model, there are 3 first order conditions, two with respect to the control variables (L and K) and one with respect to the Lagrangian multiplier. There would be more first order conditions if there were more control variables as with 2 kinds of labor or legal constraints on the possible modes of production.

$$d\mathcal{L}/dL = eL^{e-1}K^f - \lambda(w) = 0 \tag{4.2}$$

$$d\mathcal{L}/dL = fL^eK^{f-1} - \lambda(r) = 0 \tag{4.3}$$

$$d\mathcal{L}/d\lambda = (C - wL + rK) = 0 \tag{4.4}$$

To find the input demands of the firm, we follow the same steps as in the previous consumer constrained optimization problems: shift the lambda terms to the righthand side of the equations and divide one equation by the other to generate:

$$eL^{e-1}K^f/fL^eK^{f-1} = w/r$$

Which simplifies to:

$$eK/fL = w/r$$

If we want the firm's demand for labor, we solve this equation for K and then substitute that result into the constraint (the derivative of the lambda term, equation 4.4)

$$K = (w/r)\ (f/e)\ L$$

Thus, $C = wL + r\ (w/r)\ (f/e)\ L$

Reversing the sides and factoring yields:

$$L\ (w + w(f/e)) = Lw(1+f/e) = C$$

Solving for L yields an expression for the firm's demand for labor:

$$L^* = (C/w)(1/(1+f/e)) = [e/(f+e)]\ [C/w] \tag{4.5}$$

Notice that this expression looks just like the expression that we found for the consumer demand function, but in this case, it characterizes this firm's demand for labor for a given expenditure on inputs (C). A similar result can be obtained for the firms demand for capital.

$$K^* = [f/(f+e)]\ [C/r] \tag{4.6}$$

Notice also that the pattern of input demand is determined by the relative productivity of the inputs (as reflected in their respective exponents). The quantity of an input that is demanded also varies with the amount that the firm plans to spend on all of its inputs (C), and input prices (the wage rate

(w) and the cost of capital (r)). Input prices are in the denominator and so demand curves for inputs are downward sloping as true of consumer demand functions based on utility functions from this family of functions.

However, the cost function that we need describes costs in terms of outputs, rather than inputs per se. What we have at this point is the ability to describe outputs in terms of expenditures on inputs. If we know how much money is spent on all inputs, we also know how many of each of the inputs are employed. This allows us to determine how much output is produced using the production function.

The output associated with a given level of expenditure on inputs can be determined by substituting the ideal input quantities into the production function. Recall that the firm's output is $Q = L^e K^f$

Our two input demand functions allow the firm's output to written as a function of production costs by substituting the two input demand functions into the production function:

$$Q = \{[e/(f+e)]\ [C/w]\}^e\ \{[f/(f+e)]\ [C/r]\}^f$$

This characterizes output in terms of overall expenditures on inputs, their productivity (as characterized by the exponents) and input prices. This expression can be solved for C (the total cost or expenditures on inputs). Begin by factoring it out of the righthand side expression.

$$Q = C^{e+f}\ \{[e/(f+e)]\ [1/w]\}^e\ \{[f/(f+e)]\ [1/r]\}^f$$

Next solve for C as a function Q and the other parameters of the choice setting. This allows Cost (C) to be written as a function of output (Q), which is what we need for a total cost of production function.

$$C^{e+f} = Q\ /\{[e/(f+e)]\ [1/w]\}^e\ \{[f/(f+e)]\ [1/r]\}^f$$

Now take the e+f root of both sides to characterize total cost as a function of output levels, technology (represented here as the exponents), and input prices.

$$C^* = \{Q\ /\{[e/(f+e)]\ [1/w]\}^e\ \{[f/(f+e)]\ [1/r]\}^f\ \}^{1/(e+f)} \qquad (4.7)$$

This is one of the possible characterizations of C*, and the firm's cost function.

An Alternative Derivation: Minimizing the Cost of Given Outputs

Another way to derive a firm's cost function is to use the "dual" of the firm's optimization problem. In some cases, this yields a cleaner and more direct result. The "dual" choice problem requires us to minimize cost (expenditures on inputs) subject to producing a given output Q. Essentially, the dual just reverses the objective function and constraint. The new Lagrangian function is:

$$\mathcal{L} = - wL + rK + \lambda(Q - L^eK^f) \qquad (4.8)$$

(I've again used a script L ($\mathcal{L}$) for the Lagrangian equation, because L is being used for the quantity of labor employed producing the good of interest.)

Again, there are 3 first order conditions (first derivatives being set equal to zero), two with respect to the control variables (L and K), and one with respect to the Lagrangian multiplier. The first two are very similar to those we derived before, but the last is quite different.

$$d\mathcal{L}/dL = w - \lambda(eL^{e-1}K^f) = 0 \qquad (4.9)$$

$$d\mathcal{L}/dL = r - \lambda(fL^eK^{f-1}) = 0 \qquad (4.10)$$

$$d\mathcal{L}/d\lambda = (Q - L^eK^f) = 0 \qquad (4.11)$$

Notice that the only major difference in the first order conditions is the derivative with respect to the Lagrangian multiplier, λ.

To derive the firm's total cost function, very similar steps are undertaken to those in the previous derivation, but in this case, solutions will be in terms of output (Q) rather than expenditures on inputs (C).

Shifting the lambda terms in the first equations to the right and dividing yields:

$$w/r = eK/fL$$

which geometrically can be interpreted as the tangency condition(s) of figure 4.1. If we again focus on labor initially, we want to specify capital in terms of labor, which is.

$$K^* = (fw/er)\ L$$

Substituting this into the production function and solving for L, again takes a few steps:

$$Q = L^eK^f = L^e[(fw/er)\ L]^f$$

L can be factored out of the righthand expression:

$$Q = L^{e+f}(fw/er)^{f}$$

We can then solve for L* in terms of Q:

$$L^* = [Q\ (er/fw)^{f}\ ]^{1/e+f} \qquad (4.12)$$

Recall that $(x/y)^{-e} = (y/x)^{e}$, thus the ratio inside the brackets "flips" as one derives L*. This characterizes the **demand for labor as a function of output**, productivity (again indicated by the exponents) and the price of labor and capital (w and r).

We can solve for K* in a similar way. Isolating the L (instead of K) yields:

$$w/r = eK/fL$$

which yields

$$L = (e/f)K\ (r/w)$$

Substituting this into the constraint yields:

$$Q = L^{e}K^{f} = [(e/f)K\ (r/w)]^{e}\ K^{f}$$

The K can be factored out:

$$Q = K^{f+e}\ [(er/fw)]^{e}$$

Solving for K yields

$$K^* = [Q\ (fw/er)^{e}\ ]^{1/f+e} \qquad (4.13)$$

This characterization of K* is the **firm's demand for capital as a function of output**, input prices, and their productivities. The cost function can now be written in terms of the optimal quantity of labor and capital for various quantities of output:

$$C = wL^* + rK^* = w\ [Q\ (er/fw)^{f}\ ]^{1/e+f} + r\ [Q\ (fw/er)^{e}\ ]^{1/f+e} \qquad (4.14)$$

This is somewhat more intuitive expression for the firm's total cost of production than obtained in the first derivation (although they are mathematically equivalent to one another). Note that the first term is the firm's expenditure on labor and the second is the firm's expenditure on capital used in production in the **optimal amounts for the output quantity of interest**. In both cases, production costs vary with technology (the size of the exponents) and input prices. Costs clearly rise with input prices (recall that the exponents are less than 1), and costs tend to fall as the sum of the exponents fall.

Both derivations of the firms total cost function are characterizations of **long run total costs**, because the firm has been assumed to be able to vary all of its inputs.

Connecting Up the Theory of Production with the Theory of Supply

Equation 4.14 can be used to characterize this firm's supply curve. Before doing so, it will be useful to simplify the notation a bit by grouping terms and naming the groups. (This reduces the chance that a term will be dropped during the derivation.) Define, term m^L as $m^L = (er/fw)^f$, $m^K = (fw/er)^e$, and define term α as $\alpha = 1/(f+e)$. These two groupings allow the cost function (equation 4.14) to be written as

$$C = w\,(Qm^L)^{\alpha} + r\,(Qm^K)^{\alpha}$$

These "new" variables do not change when we calculate profit maximizing outputs, since they do not include quantity as a variable—but they would change if wages, capital rental costs or technology change. The simpler notation reduces the likelihood of algebraic mistakes in deriving the supply curve. After our derivation of the supply curve is complete, we can substitute the "real" expressions behind the three new terms back into the equation worked out to see how these variables affect the firm's supply decision.

The firm's profit maximizing output is calculated in the same manner as in chapter 3. Profit is total revenue (PQ) less total cost, now written as $C = w\,(Qm^L)^{\alpha} + r\,(Qm^K)^{\alpha}$.

$$\Pi = PQ - w\,(Qm^L)^{\alpha} - r\,(Qm^K)^{\alpha} = PQ - wQ^{\alpha}(m^L)^{\alpha} - rQ^{\alpha}(m^K)^{\alpha} \qquad (4.15)$$

Differentiating with respect to Q yields:

$$P - \alpha\,wQ^{\alpha-1}(m^L)^{\alpha} - \alpha\,rQ^{\alpha-1}(m^K)^{\alpha} = 0 \quad \text{at } Q^*$$

The first term (P) is marginal revenue, the others are the firm's marginal cost. The individual terms show the part of marginal cost attributable to labor costs and to capital costs. Keep in mind that we **have derived long run total cost** rather than short run marginal cost, because we are assuming that both labor and capital can be varied in the period of interest. So, this first-order condition characterizes the firm's long run profit maximizing output. Short run cost and supply would be derived by holding the quantity of capital or some other input(s) constant, which would transform the two-input case into a variation of the one input case modelled in the first part of this chapter.

One can solve for Q* (the profit maximizing output) by shifting P to the righthand side, multiply both by negative 1 and factoring.

$\alpha\, wQ_{\alpha-1}\,(m_L)^{\alpha} + \alpha\, rQ^{\alpha-1}\,(m_K)_{\alpha} = P$

$Q^{\alpha-1}\,[\alpha\, w\,(m^L)^{\alpha} + \alpha\, r\,(m^K)^{\alpha}\,] = P$

$Q^{\alpha-1} = P\,/[\alpha\, w\,(m^L)^{\alpha} + \alpha\, r\,(m^K)^{\alpha}\,]$

$$Q^* = \{\, P\,/[\alpha\, w\,(m^L)^{\alpha} + \alpha\, r\,(m^K)^{\alpha}\,]\}^{1/(\alpha-1)} \qquad (4.16)$$

Equation 4.16 is the firm's long-run supply function. Notice that this **firm's long run supply curve** is upward sloping in price. However, the quantity is reduced when wages or rental cost of capital increase through effects on m^L and m^K . —although fully determining this requires checking the derivatives of m^L and m^K to know for sure.

If there are M firms in the market with similar cost functions, then the market supply function (or curve) is simply M times that of the typical or average firm, which is.

$$Q^S = MQ^* = M\,\{\, P\,/[\alpha\, w\,(m^L)^{\alpha} + \alpha\, r\,(m^K)^{\alpha}\,]\}^{1/(\alpha-1)} \qquad (4.17)$$

As before, if firms are not identical or very similar, market supply requires adding up the supply functions of each firm, rather than simply multiplying one of the supply curves by the number of firms in the market. The assumption that suppliers have identical cost functions is sometimes called the Marshallian assumption about competitive markets, as previously mentioned.

V. Production Models with More General Families of Functions

As true of many areas of economics, using more abstract families of functions to ground one's model often makes deriving implications of a particular type of choice setting easier and the results more general. This is true of the theory of production. The easiest applications of these methods are often two- or three-dimensional problems in which there is only a single "degree of freedom" because of the effects of constraints. However, the general approach can be used for any number of variables—its just that in those cases, matrix methods for derivatives need to be employed and these are rarely used in economic research because the results are often very complex, and signs obtained are usually ambiguous—not that ambiguity is never of interest. This text eschews such methods partly for this reason and partly so that more space and class time can be devoted to understanding the implications and foundations of economically relevant decisions that are neglected in other texts.

When applied to production, one can begin with the two-input case. A general family of production functions is Q=q(L, K) with positive first derivatives, negative second derivatives and positive cross partials. The cost of inputs is again C=wL+rK. Give the cost of inputs, we can rewrite the production function as:

$$Q = q(L, (C-wL)/r) \qquad (4.18)$$

Differentiating with respect to L to characterize the firm's ideal use of labor gives us:

$$\frac{dQ}{dL} = \frac{dQ}{dL} - \frac{dQ}{dK}\left(-\frac{wL}{r}\right) = 0 \qquad (4.19)$$

We can apply the implicit function theorem to characterize any variable in the first-order condition in terms of the other variables in the equation. Each of the partial derivatives of function q includes the same variables as the parent function. Thus, L* can be characterized as:

$$L^* = l(C, w, r) \qquad (4.20)$$

Once we know the ideal quantity of labor for a given expenditure (C) on inputs, we can substitute that back into the production function to determine the output produced for a given expenditure and input prices.

$$Q^* = q(L^*, (C-wL^*)/r) \qquad (4.21)$$

Notice that if we subtract either side from the other, we get another zero function.

$$Q^* - q(L^*, (C-wL^*)/r) = 0 \qquad (4.22)$$

This allows us to apply the implicit function theorem again. In this case, our zero function is of the form h(Q, C, r, w) = 0, which we can used to characterize total production costs, C:

$$C = c(Q, w, r) \qquad (4.23)$$

This is the firm of interest's long run total cost function. Notice that the cost functions for the multiplicative exponential family of production functions took this form, but it included indicators for technology. To include technology using this approach, a variable such as "T" would be included as an exogenous variable in the production function.

The profit maximizing output, Q*, can be determined in the usual way, as worked out in chapter 3. What is of interest here is that a firm's long run cost function is generally a function of output and input prices. Input prices affect production costs both because of their effects on input costs and effects on the mix of inputs employed.

VI. A Few Conclusions

The profit maximizing model has clear implications for the production processes that firms will adopt—namely they will be the least cost method for producing the goods or services that they plan to sell. What the geometry and calculus of production functions demonstrate is that input prices matter and also a firm's ability to substitute one input for another. When substitution is possible, profit maximization implies that the marginal rate of substitution among inputs (which is determined by the technology of production) will equal the relative prices of the inputs used in production. Any deviation from that relationship would imply that profits are not being maximized. Moreover, this will be true if profits are maximized even if the firm owners have not explicitly taken marginal rates of substitution into account. It is a property of profit maximization when inputs are substitutable.

There are cases in which substitutability is limited and also cases in which a particular input such as air or water may be freely available. In the former case, the effects of input prices on input mixes are limited—and in extreme cases may not exist. In the latter cases, free inputs will be used at the level that maximizes their contribution to production. That is generally at the rate where their marginal product falls to zero. Freely available air and water, for example, are used in several ways in many industrial processes. They may serve as coolants, they may be incorporated into the product itself, or they may be used as a convenient way to dispose of waste products. In all cases where a subset of inputs is freely available, corner solutions rather than the tangency solutions of the diagrams and calculus models will be evident for those inputs, but not others that are costly.

Nonetheless, although there are obvious exceptions and limitations to the results obtained from modelling production processes where inputs are costly and substitutions among inputs are possible, these are not unrealistic assumptions about most production processes. Although bicycles may need two wheels, the wheels can be made from a variety of materials—thus the manufacturing of wheels can be modelled in the way that that neoclassical economics suggests, even if those models have limitations regarding bicycles per se. Moreover, deriving cost functions in cases in which substitutability tends to be limited is easier and more direct than in cases where substitutability is not possible.

The families of production functions used to model production provide clear foundations for the cost functions that ground the models of market supply developed in chapter 3. One does not have to ignore production choices to model supply. An integrated model can be developed fairly easily as demonstrated in this chapter. The derivations make it clear why cost functions should include input prices and technology as variables. That inclusion, together with the results from chapter 3, make it clear why changes in such factors affect market supply, and as demonstrated in the next chapter, why they affect market prices for final goods.

VII. Selected References

Cobb, C. W., & Douglas, P. H. (1928). A Theory of Production. *The American Economic Review*, *18*(1), 139-165.

G. S. Becker, "A Theory of the Allocation of Time," Econ. J., Sept. 1965, 75, 493- 517.

Ferguson, C. E. (2008). *The neoclassical theory of production and distribution.* Cambridge UK: Cambridge University Press.

Georgescu-Roegen, N. (1972). Process analysis and the neoclassical theory of production. *American Journal of Agricultural Economics*, *54*(2), 279-294.

Gronau, R. (1997). The theory of home production: the past ten years. *Journal of Labor Economics*, *15*(2), 197-205.

Nicholson, W. (1992). *Microeconomic Theory*. Fort Worth TX: Dryden Press.

Samuelson, P. A. (1962). Economists and the History of Ideas. *The American Economic Review*, *52*(1), 1-18.

Varian, H. R. (1992). *Microeconomic Analysis*. New York: WW Norton & Company.