Chapter 4: Production: Making Things to Sell

I. Introduction: Production for Sale

When economists talk about production, they usually refer to activities undertaken to transform a group of materials and services into others of greater value. In a commercial society, the individuals or organizations that produce goods or services usually plan to sell them to other persons, groups, or organizations. Thus, production in most economic models is not just adding value or providing services but making stuff and providing services that others in their communities or trading networks are willing and able to purchase.

Of course, not everything is produced for sale, nor is everything sold in markets newly produced for sale. Nonetheless, the supply of many products and most services is produced shortly before they are brought to market. Characterizing how such products are produced is the main focus of this chapter.

Household Production

This focus is not meant to imply that the term "production" is never used to describe the manner in which goods that are not for sale are brought into existence. For example, that term is also used by economists to describe the activities of persons who raise some of their own vegetables in gardens, bake bread, or construct sheds to store tools in their backyards, without any intent to sell them. Such activities are often termed **household production** (Becker, 1965).

In the periods and places before commercial societies emerged, a good deal, perhaps most, of a family's time, knowledge, and materials were devoted to such productive activities. Foodstuffs were homegrown or the product of hunting and gathering in nearby forests. A good deal of clothing was homemade. Thread was home spun, and cloth created on looms at home or by nearby weavers. Cooking was conducted over fires fueled by wood harvested from nearby woodlands, and so forth. Although a lot of time was spent producing goods and services in the period before commerce emerged, little or none was produced for sale.

Nonetheless, household production can be modeled using rational-choice models and historically played a role in the emergence of markets. Household production is a utility maximizing choice (an allocation of time), rather than a profit maximizing one. Moreover, household production remains an important enterprise today, even if less so than in past millennia. Most of what we bring home from grocery and building stores is used in household production—as, for example, the "inputs" of bread, peanut butter, and jelly, together with labor, and capital in the form of knives, plates, and napkins are used to construct a peanut butter and jelly sandwich. Additional equipment and more labor are used to clean up afterwards. Lumber purchased by consumers may be transformed into bookshelves or tables using one's own labor and capital (saws, measuring tapes, hammers, clamps, and so on).

Where household production takes place in a commercial society, it indirectly affects prices through demands for inputs and effects on the markets for final products. Conversely, market prices affect decisions to engage in household production. Household production is undertaken because the quality of the final product produced at home is deemed superior to that which can be purchased in markets or less costly in terms of time and money.

The main difference between household production and production for sale in markets is a difference in objectives. In household production, the aim is to increase utility either because the production process itself is valued, or indirectly, because the outputs are regarded to be more pleasing or less expensive than substitutes that could have been purchased in markets.

Production for sale takes place when the net income from producing and selling goods yields more utility than engaging in household production. This is normally because specialization and team production decrease the cost of producing goods and services relative to household production. Many of the products that are readily available in markets cannot be produced at home.

Intermediate cases also exist in which a person or group of persons produce some things for themselves (as with vegetable gardens, hunters, or carpenters) and sell part of the things produced to their acquaintances or in local markets. Such intermediate cases were likely to be the first commercial producers. It was through household production that many production methods were worked out and through comparative advantage that gains to trade and a degree of specialization emerged among households within a village or region.

As markets extended, such intermediate cases gradually became increasingly commercial—as families devoted more and more of their resources to production for sale, and less and less for household production. That, for example, accounts for the transition from subsistence farming to family farms, and subsequently to corporate farms.

Market Supply and Production

Not everything that is sold is produced by firms and not everything that is produced has recently been produced. Preexisting things may be sold. This includes products of nature and previously produced goods. Land, buildings, classic artwork, used cars, used books, and used cell phones, for example, are routinely resold. These are often non-trivial markets, as, for example, more used cars are sold in a given year than new ones. Such goods have an opportunity cost—the presently most highly valued uses—but do not have production costs when they are sold.

So, there are goods that are produced and not sold, and products that are sold but not recently produced. (Indeed the latter are not counted in gross domestic product statistics.) However, most of the products sold in contemporary markets are newly produced.

The supply functions of the goods analyzed in chapter 3 were all for goods that were newly produced. They were produced because the value added through the productive efforts of firms increase their value sufficiently that profits could be realized by selling them. If sufficient value was not added to the inputs used, the cost of the inputs would exceed the market value of the products produced, and losses would be realized instead of profits by selling them. Such firms and production processes would not long survive.

Larger commercial organizations became commonplace in the twentieth century during the period in which dense and ubiquitous commercial networks emerged. Very large commercial organizations are relatively new—less than a couple of centuries old. Yet, such organizations, while important, have not totally displaced small businesses. Family-based and other relatively small firms are still commonplace and account for roughly half of the jobs in today's commercial societies.

It is the market relationships of late nineteenth and early twentieth century commercial societies in the West that neoclassical economics emerged to explain. Until then, farming was the largest economic enterprise and working on farms and closely related businesses were the main source of employment. It was largely during the nineteenth century that production within both relatively small and large organizations devoted to selling their output(s) became commonplace and household production for own use diminished in importance. More and more persons "hired themselves out for wages" at relatively large industrial and retail corporations. They used the proceeds to purchase both the necessities of life and other goods consumed for pleasure.

Together, the neoclassical economic theory of supply and production characterize the processes through which new things or services are created and sold in markets. They do so in a somewhat abstract and general fashion. They do not focus on the details of marketing and production, but rather common features of those activities. The neoclassical theory of production models the choices about production that generate a firm's cost functions. Their cost functions, in turn, largely determine a firm's output decisions. Only profitable outputs are intentionally produced. And to the extent that firms successfully maximize their profits, only least-cost methods of production are used.

II. The Simplest Integrated Model of Production and Supply

The simplest methods of production use only a single variable input such as labor. Other inputs may be fixed in some way, either by nature, as with gathering fruit from wild growing fruit trees, or because the other factors are generally held constant as production increases, such as might be said of an ax used to cut some trees down to build a fire, a house, or clear a pasture. One may need an ax to produce firewood, but after an individual has an ax, the rest is all labor.

The production function in such cases can be modeled with a one-dimensional function such as $Q = q(L)$ where function q is assumed to be strictly concave—e.g., to exhibit diminishing returns over the full or relevant domain of production. Total variable costs would simply be $C = wL$, w is the wage rate (or opportunity cost) of labor. Production costs in a commercial society include all expenditures on inputs. (For household production, the cost of using labor in a particular activity is its opportunity cost in other uses.)

The cost functions that we used in chapter 3 were all functions of the firm's output, a specification that was important for the firm's decision about how much output to produce and bring to market. To create a cost function in terms of output rather than inputs, we need to determine the relationship between output and inputs. If, for example, the firm's production function is

$$Q = aL^b, \tag{4.1}$$

with b<1, then the inverse function—the labor required to produce a given output, is the solution for L as a function of Q, which a bit of algebra finds is:

$$L = (Q/a)^{1/b}. \tag{4.2}$$

Given that relationship we can write the cost function as a function of output by substituting the inverse function that describes how much labor is being used as a function of quantity into the firm's cost function. In the first case,

$$C \; = \; wL \; = \; w\,(Q/a)^{1/b}. \tag{4.3}$$

Algebraically, this cost equation looks a bit complicated, but notice that it is very similar to the simplest of the exponential cost functions that we used in Chapter 3 (e.g. $C = aQ^b$). The firm's profit function is:

$$\Pi \; = \; PQ \; - \; w(Q/a)^{1/b}. \tag{4.4}$$

To characterize the firm's optimal output, differentiate the profit function with respect to Q, and set the result equal to zero. That relationship characterizes, Q*, the output that maximizes profit. In the simple case being worked through, this yields

$$P \; - \; w\left(\frac{1}{b}\right)\left(\frac{Q}{a}\right)^{\left(\frac{1}{b}\right)-1} \; = \; 0 \tag{4.5}$$

The first term is marginal revenue (P) and the second more complex term is marginal cost. The firm's supply function is found by solving that relationship for Q*, which requires a bit of algebra. First, we rewrite equation 4.5, then isolate the Q.

$$P \; = \; w\,(1/b)(Q/a)^{(1/b)-1} \;\rightarrow\; bP/w \; = \; (Q/a)^{(1-b)/b} \rightarrow \; (bP/w)^{b/(1-b)} = (Q/a)$$

which implies that

$$Q^* \; = \; a(bP/w)^{b/(1-b)}. \tag{4.6}$$

This is the firm's supply function given the production function assumed. Note that supply rises with price and falls as input prices increase.

Deriving the supply function in the general case of production with a single input follows the same steps, although with less algebra. If the production function is

$$Q = q(L), \tag{4.7}$$

then the inverse of the production function is characterized using the implicit function theorem on $0 = Q - q(L)$. That function is written as $L = q^{-1}(Q)$, where the (-1) denotes an inverse function, rather than an exponent of the q function. The firm's cost function is simply:

$$C = w\, q^{-1}(Q) \qquad (4.8)$$

and its profit function (if a price taker) is simply

$$\Pi = PQ - w\, q^{-1}(Q). \qquad (4.9)$$

Notice that this is identical with the model of the firm's choice used to generate the supply function in chapter 3, except the cost function is now explicitly derived from the firm's production process—rather than simply assumed—and the cost function now explicitly takes account of the input price (here the wage rate of labor). To characterize the firm's supply curve we, again, differentiate its profit function with respect to Q and set the result equal to zero.

$$d\Pi/dQ = P - w(dq^{-1}/dQ) = 0 \equiv H. \qquad (4.10)$$

The first term (P) is the firm's marginal revenue and the second [$w(dq^{-1}/dQ)$] is its marginal cost.

To characterize the firm's supply function, we appeal to the implicit function theorem, which implies that Q* (the quantity that satisfies the first order condition) can be written as:

$$Q^* = s(P, w). \qquad (4.11)$$

This is the firm's supply function.

The implicit function differentiation rule can be used to find the effects of price and wage rates on the firm's output. The first order condition is used as the relevant zero function.

$$dQ^*/dP = (dH/dP)\,/\,(-\,dH/dQ^*) = 1/-(\,d^2\Pi/dQ^2) > 0 \qquad (4.12a)$$

$$dQ^*/dw = (dH/dw)\,/\,(-\,dH/dQ^*) = \frac{-\frac{dq^{-1}}{dw}}{-\left(\frac{d^2\Pi}{dQ^2}\right)} < 0 \qquad (4.12b)$$

The denominator has been written in terms of derivatives of the profit function to simplify and shorten the derivation. The second derivative of the profit function is negative if it is strictly concave, so the denominators of both of these derivatives are positive (because of the leading negative sign).

The numerator of the derivative with respect to price is simply 1, which is positive. So, the overall effect of an increase in price on the quantity supplied is positive, as in the concrete functional form case. The numerator of the derivative with respect to input price w is the negative of the slope of the inverse production function. The inverse production function and production function have the same slope, which is positive in this case because marginal product has been assumed to be greater than zero over the range of interest. Thus, the overall effect of an increase in wage rates on the quantity supplied is negative. An increase in wage rates shifts the supply function (curve) back to the left.

These are quite general relationships that follow whenever the production function is subject to diminishing returns overall or within the domain of interest.

In terms of the mathematics, most of the complexity is generated by the necessity of generating a cost function that specifies a firm's total cost as a function of output levels. This method of deriving a firm's supply function is most similar to that used in chapter 3 to characterize a firm's supply decision.

Note that we could have reversed the order of the math in cases where production involves just one input. Profit could have been written in terms of labor usage as

$$\Pi \ = \ Pq(L) - wL. \tag{4.13}$$

The ideal level of labor is that which maximizes profit, which can be determined by differentiating the profit function with respect to L and setting the result equal to zero. This yields:

$$P\ (dQ/dL) \ - \ w \ = \ 0. \tag{4.14}$$

This first order condition implies that the firm will hire labor up to the point where its marginal revenue product, $P\ (dQ/dL)$ equals the wage rate (w). L* can be characterized using the implicit function theorem as

$$L^* = \ g(P, w). \tag{4.15}$$

This is **the firm's demand for labor** which varies with its price (w) and the selling price of the output (P).

The output associated with that choice can be characterized using the production function, as with

$$Q^* = \ q(L^*). \tag{4.16}$$

Given the variables in L*, the implicit function theorem implies that Q* can be written as

$$Q^* = \ s(P, w) \tag{4.17}$$

as before.

Unfortunately, these very clean and general derivations can only be undertaken for one input production functions With multi-input production, the derivations are more complex, but yield fundamentally similar results as developed below.

Nonetheless, the result that firms hire inputs up to the point where their marginal product times the price of the output equals the price of the input, is quite general and a very useful rule of thumb to keep in one's mind.

III. The Geometry of Production Requiring More than One Input

The geometry used to model production choices is similar to that used to characterize a consumer's choices among several goods and services. However, instead of attempting to maximize utility given a budget constraint, firms attempt to maximize production for given expenditures on inputs. Another difference between consumer and firm choices is the assumption that a firm's expenditures are in a sense unlimited because of access to capital markets. Firms, thus, have a broad range of expenditures on inputs that they can undertake. The expenditure level chosen by a profit maximizing firm is the one that maximizes profits as characterized in chapter 3. In this chapter, the determination of the profit maximizing output level and expenditures on inputs are simultaneously chosen and explicitly modelled.

The isocost lines characterize the various combinations of inputs that can be purchased for a given expenditure (e.g. cost). Each iso-cost line represents a particular total expenditure on two goods (L and K) at their given prices (w and r).

Firms naturally attempt to minimize the cost of producing each possible output level. This occurs at **tangency points between the isocost and isoquant lines,** three of which are illustrated. There is a total cost (total expenditure on inputs) and an output level at each point of tangency. Together the tangencies associated with different outputs, characterize the cost (minimum cost) of producing those outputs—which is to say they characterize the firm's **total cost function**. The production process adopted is the one that minimizes the cost of the profit-maximizing output for the firm, given input prices and the selling price of the final good. The diagram shows that it costs amount C_1 to produce quantity Q_1, amount C_2 to produce quantity Q_2, and amount C_3 to produce quantity Q_3. Thus (C_1, Q_1), (C_2,Q_2) and (C_3, Q_3) are all points on the firm's cost function.

That output of the firm is not determined by the diagram drawn, but the cost function characterized by the diagram. The cost function is used by the firm to determine its profit maximizing output, Q^*. The production method used is the one on the Q^* isoquant that is tangent to the iso-cost line. It is the least cost method of producing Q* units of the good. The method of production is indicated by the combination of inputs that the firm uses.

Figure 4.1 Relationship between Output, Cost and Input Mix for a Price-Taking Firm

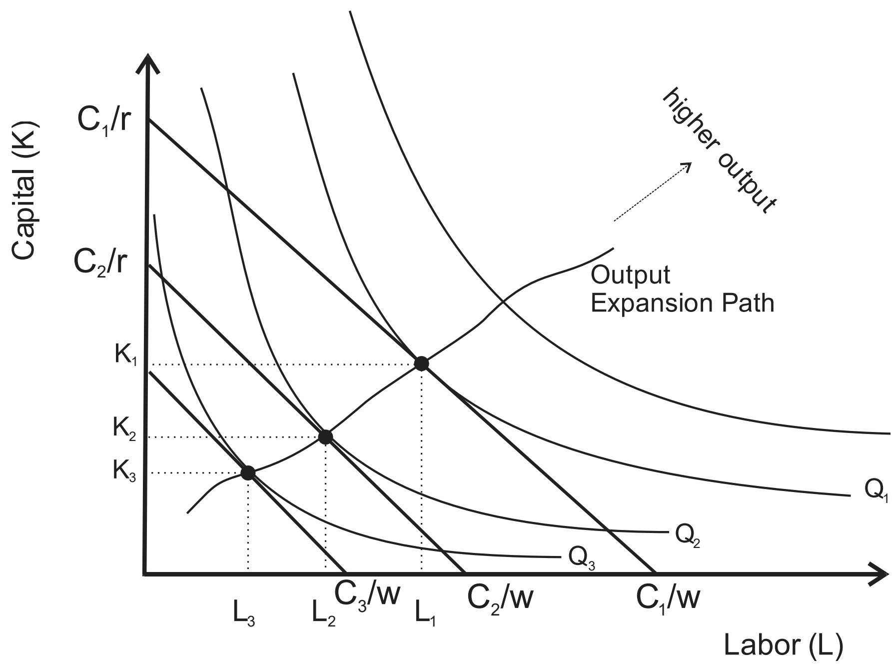

Figure 4.1 thus shows in principle how firms produce their products when a particular multi-input technology or combination of multi-input technologies is used.

Price-taking firms adjust their production methods to take account of the prevailing input prices (w, wages, and r, the rental cost of capital). The output produced reflects both input prices and the prevailing price of its output (P).

If any of the input prices change, the optimal combinations of inputs used to produce its output will also change, which alters its total and marginal cost functions. Geometrically, such changes alter the slope of a firm's isocost lines.

For example, given the assumed C-shaped isoquants, when the price of labor increases, firms will use more capital and less labor. This adjustment increases both total and marginal production costs. In the diagram, this occurs because as w increases the slopes of the iso-cost lines change and the least cost method of producing a particular output such as Q* tends to increase. That is to say, the tangency point for each isocost curve occurs at a lower isoquant.

If we knew the numbers associated with the iso cost curves and isoquants, we could have precisely characterized total cost and marginal cost. This would require knowledge of both the production function used and input prices.

The qualitative results generated by diagrams similar to figure 4.1 provide **a number of insights**. First, profit maximizing firms produce with input mixes that equate **marginal rates of technical substitution (the slope of an isoquant) with their relative prices (the slope of an iso cost line).**

Second, technologies that exhibit constant return to scale tend to have linear output expansion paths. In contrast, technologies that exhibit economies of scale often require the use of more capital-intensive methods as output expands. For example, in the nineteenth century, cost-minimizing production of a number of products required more capital as more output was produced—as with the production of steel, many chemicals, and electricity. Such cases are even more commonplace in the twentieth and twenty-first centuries. Large firms making use of capital-intensive production have output-expansion paths that become increasingly steep as output increases.

Third, the **distinction between the long run and short run adjustments** can also be explained by the same model. If one (or more) of the factors of production is fixed in the short run, the cost of increasing output tends to be higher than when it is not. Normally the factor that is most time-consuming to adjust in the short run is capital. Erecting new factories and equipping them usually takes longer than adding a few more people to the production process. (Although it should be noted that some types of labor may also take a long time to change—as with some types of firm-specific human capital and others that require long training periods, such as doctors, lawyers, software management, and economists.)

In the usual case, this means that short run supply decisions cannot use capital-labor substitution to economize on rising labor costs in the short run, although they can do so in the long run. Cases in which specialized labor cannot be varied in the short run can be represented in a similar way. Hospitals may be able to add medical equipment more rapidly than more doctors. Both problems for firms imply that reaching a higher isoquant will require greater expenditures on inputs in the short run than in the long run. Thus, short run marginal cost curves tend to be steeper than long run marginal cost curves.

Diagrams of this sort can also be used to characterize production functions that are actually quite common, but that do not lend themselves to the tools of calculus. Namely production processes in which various **impossibilities of substitution** occur (**Leontief production**

functions). For example, production may require that particular ratios of inputs be used to produce the desired output. Bicycles, for example, always have two wheels (here partly by definition, but also partly because of the physics of riding them). If the price of wheels or hubs changes, no substitution (holding quality constant) of, say, pedals or seats for wheels is possible. Each bicycle needs two wheels, two hubs, and two tires. All other possible uses of inputs are wasteful

Deriving cost functions in cases in which substitutability tends to be limited is easier and more direct than in cases where substitutability is relatively easy. When substitution among inputs is impossible, unit production costs are simply the sum of the necessary inputs. Total costs are simply the number of units in cases time quantity if production exhibits constant returns, QC. If there are economies or diseconomies of scale, production costs may be modelled as $(QC)^b$ with b greater or less than one according to whether there are economies or diseconomies of scale.

There are also cases in which a particular input such as air or water may be freely available. In such cases, the freely available resources such as air and water are used as long as their marginal product is greater than zero. They may serve as coolants, they may be incorporated into the product itself, or they may be used as a convenient way to dispose of waste products. In cases where a subset of inputs is freely available, corner solutions rather than the tangency solutions of the above diagram will be evident for the free inputs, but not others that are costly.

The calculus approach used throughout the rest of this chapter sheds useful light on aspects of production where substitution among inputs and associated methods of production is possible and all inputs are scarce and therefore costly to employ. Production is a more flexible process in such cases, and the choice of production methods is more complicated.

IV. The Calculus of Production Using Concrete Multi-Input Production Functions

Just as the geometric representation of a firm's multi-input production decisions is similar to that used for modeling consumer choices over many goods, the calculus-based model of a firm's multi-input decision is very similar to that used to characterize consumer choices among different combinations of goods. Even the family of functions focused on tend to be similar. Cobb-Douglas and multiplicative exponential functions are among the most common families of functions used.

Unfortunately, the results are somewhat more difficult to derive and more complex for production than they are for consumption. This occurs mostly because there are two stages to the choices modeled. First, the production stage is used to characterize a cost function. And second, a supply stage in which the cost function is used to derive a supply function is developed—more or less as done in chapter 3.

As in Figure 4.1, we'll focus on a two-input production process, where the inputs are labor and capital. The cost of the products or services produced are determined by market prices for inputs and the use rates of those inputs: $C = wL^* + rK^*$, where L^* and K^* are the usage of labor and capital to produce the profit maximizing output in the least cost way possible under the existing technology... As in the geometric case, a firm's production cost depends on factor prices (w and r) and the production function used, because these affect the combinations of inputs used to produce the products to be sold.

Suppose also that the firm's output from the use of labor and capital is $Q = L^e K^f$, where both e and f are greater than 0, but less than 1. The "technology" of production is represented both by the inputs required and the exponents of the production function assumed.

The production process is regarded to be exogenous (beyond the control of the firm) and constant during the period of interest (sometimes called the planning horizon). The prices of inputs (w and r) are also parameters of the firm's choice setting when firms are price takers in input markets.

The assumption that the sum of the exponents is less than 1 and greater than zero implies that each factor of production contributes to output but also exhibits diminishing marginal returns. The overall production process also exhibits **diminishing marginal returns**. In such cases, the associated marginal cost function is upward sloping—as developed below. If their sum is exactly 1, this would be a **Cobb-Douglas** production function, and the overall production process would exhibit **constant returns to scale**.

The firm's cost function can be derived either by minimizing the cost of a given output or by maximizing the output achieved for a given cost. These approaches can be regarded as the "**dual**" of the other. The results should be identical to one another, and so the best approach is the one that appears most likely to be error free.

We'll first use the maximizing output for a given cost approach, because this is similar to that used for deriving consumer demand and is, in a sense, the "natural" way to characterize a firm's cost function. Perhaps surprisingly, the results are more difficult to derive in this way and more difficult to interpret than the less intuitive solution to the dual problem—as we will see.

Both derivations of optimal production methods undertaken in this section are similar to those of the consumer choice problem analyzed in chapter 2. They both characterize a firm's demand for inputs for given expenditure on inputs. However, the objective function that we use to model volition is the maximization of output for given expenditures or the minimization of the cost of a given output level rather than utility.

Maximizing Output for a Given Cost

The Lagrange approach is a bit easier than the substitution method in most cases in which a model is based on concrete functional forms. We'll assume that the production function is a multiplicative-exponential function, and that the direct cost considerations are linear, as in figure 4.1. However, as usual, the individual equations are more difficult to interpret than those obtained using the substitution method. Since L is being used to characterize the quantity of labor used in production, we'll again use a "script L" for the symbol representing the Lagrange function ($\mathcal{L}$).

The Lagrangian equation associated with maximizing the output achieved at a given cost is:

$$\mathcal{L} = L^e K^f + \lambda(C - wL - rK) \qquad (4.18)$$

As in the consumer choice model, there are 3 first order conditions, two with respect to the control variables (L and K) and one with respect to the Lagrangian multiplier. There would be more first order conditions if there were more control variables as with 2 kinds of labor or legal constraints on the possible modes of production.

$$d\mathcal{L}/dL = eL^{e-1}K^f - \lambda(w) = 0 \qquad (4.19a)$$

$$d\mathcal{L}/dK = fL^eK^{f-1} - \lambda(r) = 0 \qquad (4.19b)$$

$$d\mathcal{L}/d\lambda = (C - wL - rK) = 0 \qquad (4.19c)$$

To find the input demands of the firm, we follow the same steps as in the previous consumer constrained optimization problems: shift the lambda terms to the righthand side of the equations and divide one equation by the other to generate:

$$\frac{eL^{e-1}K^f}{fL^eK^{f-1}} = \frac{w}{r}$$

Which simplifies to:

$$eK/fL = w/r$$

If we want the firm's demand for labor, we solve this equation for K and then substitute that result into the cost function (the derivative of the lambda term, equation 4.19c).

$$K = (w/r)\,(f/e)\,L$$

Thus,

$$C = wL + r\,(w/r)\,(f/e)\,L$$

Reversing the sides and factoring yields:

$$L\,(w + w(f/e)) = Lw(1 + f/e) = C$$

Solving for L yields an expression for the firm's demand for labor:

$$L^* = (C/w)(1/(1 + f/e)) = [e/(f + e)]\,[C/w] \qquad (4.20)$$

Notice that this expression looks similar to the expression that we found for the consumer demand function, but in this case, it characterizes this firm's demand for labor for a given expenditure on inputs (C). It characterizes the employment of labor along a firm's output expansion path rather than its final choice of labor or its output level.

A similar result can be obtained for the firm's demand for capital.

$$K^* = [f/(f + e)]\,[C/r] \qquad (4.21)$$

Notice that the pattern of input demand is determined by the relative productivity of the inputs (as reflected in their respective exponents). The quantity of an input that is demanded also varies with the amount that the firm plans to spend on all of its inputs (C), and input prices (the wage rate (w) and the cost of capital (r)). Input prices are in the denominator and so demand curves for inputs are downward sloping, as true of consumer demand functions derived from utility functions from this family of functions and linear budget constraints.

However, the cost function that we need describes costs in terms of outputs, rather than inputs per se. What we have at this point is the ability to describe production costs in terms of expenditures on inputs. If we know how much money is spent on all inputs, we also know the quantities of each input that will be employed.

This relationship allows us to determine how much output is produced using the production function. The output associated with a given level of expenditure on inputs can be determined by substituting the ideal input quantities of labor and capital into the production function. Recall that the firm's output is $Q = L^e K^f$.

Our two input demand functions allow the firm's output to be written as a function of production costs by substituting the two input demand functions into the production function:

$$Q = \{[e/(f+e)]\,[C/w]\}^{e}\,\{[f/(f+e)]\,[C/r]\}^{f} \qquad (4.22)$$

This characterizes output in terms of overall expenditures on inputs, their productivity (as characterized by the exponents) and input prices.

This expression can be solved for C (the total cost or expenditures on inputs). Begin by factoring it out of the righthand side expression.

$$Q = C^{e+f}\,\{[e/(f+e)]\,[1/w]\}^{e}\,\{[f/(f+e)]\,[1/r]\}^{f}$$

Next solve for C as a function Q and the other parameters of the choice setting. This allows Cost (C) to be written as a function of output (Q), which is what we need for a total cost of production function.

$C^{e+f} = Q / \{[e/(f+e)]\ [1/w]\}^{e}\ \{[f/(f+e)]\ [1/r]\}^{f}$

$$C^{e+f} = Q\,/\left(\left\{\left[\frac{e}{f+e}\right]\left[\frac{1}{w}\right]\right\}^{e}\left\{\left[\frac{f}{f+e}\right]\left[\frac{1}{r}\right]\right\}^{f}\right)$$

Now take the e+f root of both sides to characterize total cost as a function of output levels, technology (represented here as the exponents), and input prices.

$$C^{*} = \left\{Q\,/\left(\left\{\left[\frac{e}{f+e}\right]\left[\frac{1}{w}\right]\right\}^{e}\left\{\left[\frac{f}{f+e}\right]\left[\frac{1}{r}\right]\right\}^{f}\right)\right\}^{1/(e+f)} \qquad (4.23)$$

This is one of the possible characterizations of C*, the minimum cost for every output level.

It is the firm's total cost function in terms of output levels, the prices of inputs, and the technology of production (the various exponents from the production function). After deriving the firm's cost function, one can characterize the firm's supply function in the same manner used in Chapter 3.

An Alternative Derivation: Duality and Minimizing the Cost of Outputs

Another way to derive a firm's cost function is to use the "dual" of the firm's optimization problem. In some cases, this yields a cleaner and more direct result. The "dual" choice problem requires us to minimize cost (expenditures on inputs) subject to producing a given output Q. Essentially, the dual just reverses the objective function and constraint. It characterizes the cost minimizing output levels for given outputs, rather than maximizing output for given expenditures on inputs. The new Lagrangian function is:

$$\mathcal{L} = wL + rK + \lambda(Q - L^e K^f) \quad (4.24)$$

(I've again used a script L ($\mathcal{L}$) for the Lagrangian equation, because L is being used for the quantity of labor employed producing the good of interest.)

Again, there are 3 first order conditions (first derivatives being set equal to zero), two with respect to the control variables (L and K), and one with respect to the Lagrangian multiplier. The first two are very similar to those we derived before, but the last is quite different.

$$d\mathcal{L}/dL = w - \lambda(eL^{e-1}K^f) = 0 \quad (4.25a)$$

$$d\mathcal{L}/dK = r - \lambda(fL^e K^{f-1}) = 0 \quad (4.25b)$$

$$d\mathcal{L}/d\lambda = (Q - L^e K^f) = 0 \quad (4.25c)$$

Notice that the only major difference in the first order conditions is the derivative with respect to the Lagrangian multiplier, λ.

To derive the firm's total cost function, very similar steps are undertaken to those in the previous derivation, but in this case, solutions will be in terms of output (Q) rather than expenditures on inputs (C).

Shifting the lambda terms in the first equations to the right and dividing yields:

$$w/r = eK/fL$$

which, again, geometrically can be interpreted as the tangency condition(s) of figure 4.1. We again initially focus on the demand for labor. In that case, we want to specify capital in terms of labor:

$$K^* = (fw/er)\,L \quad (4.26)$$

Substituting this into the production function and solving for L, takes just a few steps:

$$Q = L^e K^f = L^e[(fw/er)\,L]^f$$

Note that L can be factored out of the righthand expression:

$$Q = L^{e+f}(fw/er)^f$$

We can then solve for L^* in terms of Q:

$$L^* = [Q\ /\ (fw/er)^f]^{1/(e+f)} \tag{4.27}$$

This characterizes the **demand for labor as a function of output.** It varies with the productivity of the two inputs(again indicated by the exponents) and the prices of labor and capital (w and r).

We can solve for K* in a similar way. Isolating the L (instead of K) yields:

$$w/r = eK/fL$$

which yields

$$L = (e/f)K\ (r/w)$$

Substituting this into the constraint yields:

$$Q = L^eK^f = [(e/f)K\ (r/w)]^eK^f$$

The K can be factored out:

$$Q = K^{f+e}(er/fw)^e$$

Solving for K yields

$$K^* = [Q\ /(er/fw)^e\ ]^{1/(f+e)} \tag{4.28}$$

K* is the **firm's demand for capital as a function of output**, input prices, and their relative productivities.

The cost function can now be written in terms of the optimal quantity of labor and capital for various quantities of output:

$$C = wL^* + rK^* = w\ [Q\ /(fw/er)^f]^{1/(e+f)} + r[Q\ /(er/fw)^e\ ]^{1/(f+e)} \tag{4.29}$$

Note that the first term is the firm's expenditure on labor and the second is the firm's expenditure on capital used in production in the **optimal amounts for the output quantity of interest**. This is a somewhat more intuitive expression for the firm's total cost of production than obtained in the first derivation (although they are mathematically equivalent to one another).

In both cases, production costs vary with technology (the size of the exponents) and input prices. Costs clearly rise with input prices (recall that the exponents are less than 1), and costs tend to fall as the sum of the exponents falls.

Both derivations characterize a firm's **long-run total costs**, because the firm modelled has been assumed to be able to vary all of its inputs to either minimize the production cost of a given output or maximize the output obtained from a given expenditure on inputs.

Note that short-run cost functions can be derived in a manner similar to that of the one-input models at the beginning of the chapter, because holding either of the inputs constant, in effect, reduces the production function to a single control variable in the two input case—namely, the input that can be varied in the short run.

Connecting Up the Theory of Production with the Theory of Supply

Equation 4.29 can be used to characterize this firm's supply curve. Before doing so, it will be useful to simplify the notation a bit by grouping terms and naming the groups. (This reduces the chance that a term will be dropped during the derivation.) Define new terms m^L and m^K as $m^L = (er/fw)^f$, $m^K = (fw/er)^e$, and define term α as $\alpha = 1/(f + e)$. These two groupings allow the firm's cost function (equation 4.29) to be written as

$$C = w\,(Qm^L)^\alpha + r\,(Qm^K)^\alpha \qquad (4.30)$$

The new terms m^L and m^K make use of the fact that $(a/b)^{-c} = (b/a)^c$ to replace the labor (L) and capital (K) devisors with their multiplicative counterparts, which will simplify the calculus a bit.

The "new" variables do not change when we calculate profit maximizing outputs, because they do not include quantity as a variable—but they would change if wages, capital rental costs or technology change. The simpler notation of equation 4.30 reduces the likelihood of algebraic mistakes in deriving the supply curve. After our derivation of the supply curve is complete, we can substitute the "real" expressions behind the three new terms back into the equation worked out to see how these variables affect the firm's supply decision.

The firm's profit maximizing output is calculated in the same manner as in chapter 3. Profit is total revenue (PQ) less total cost, now written as $C = w\,(Qm^L)^\alpha + r\,(Qm^K)^\alpha$.

$$\Pi = PQ - w(Qm^L)^\alpha - r(Qm^K)^\alpha = PQ - wQ^\alpha(m^L)^\alpha - rQ^{\,\alpha}\,(m^K)^\alpha \qquad (4.31)$$

Differentiating with respect to Q yields:

$$P - \alpha wQ^{\alpha-1}(m^L)^\alpha - \alpha rQ^{\,\alpha-1}\,(m^K)^\alpha = 0 \text{ at } Q^* \qquad (4.32)$$

The first term (P) is marginal revenue, the other terms are the firm's marginal cost. The individual terms show the part of marginal cost attributable to labor costs and to capital costs. Keep in mind that we **have derived long run total cost** rather than short run marginal cost, because we are assuming that both labor and capital can be varied in the period of interest. So, this first-order condition characterizes the firm's long-run profit maximizing output. Short run cost and supply would be derived by holding the quantity of capital or some other input(s) constant, which would transform the two-input case into a variation of the one input case modelled in the first part of this chapter.

One can solve for Q* (the profit maximizing output) by shifting P to the righthand side, multiplying both sides by negative 1 and factoring.

$$\alpha w Q^{\alpha-1}(m^L)^\alpha + \alpha r Q^{\alpha-1}(m^K)^\alpha = P$$

$$Q^{\alpha-1}[\alpha w(m^L)^\alpha + \alpha r(m^K)^\alpha] = P$$

$$Q^* = \{P/[\alpha w(m^L)^\alpha + \alpha r(m^K)^\alpha]\}^{1/(\alpha-1)} \qquad (4.33)$$

Equation 4.33 is the firm's long-run supply function. Notice that this **firm's long run supply curve** is upward sloping in price, because the prevailing market price for its output is in the numerator of the lefthand side. However, the quantity is reduced when wages or rental cost of capital increase through effects on m^L and m^K. —although fully determining this requires checking the derivatives of m^L and m^K to know for sure.

If there are M firms in the market with similar cost functions, then the market supply function (or curve) is simply M times that of the typical or average firm:

$$Q^S = MQ^* = M\{P/[\alpha w(m^L)^\alpha + \alpha r(m^K)^\alpha]\}^{1/(\alpha-1)} \qquad (4.34)$$

As before, if firms are not identical or very similar, market supply requires adding up the supply functions of each firm, rather than simply multiplying one of the supply curves by the number of firms in the market. The assumption that suppliers have identical cost functions is sometimes called the Marshallian assumption about competitive markets (Marshall, 1920), as discussed in Chapter 3.

V. Production Models with More General Families of Functions

As true of many areas of economics, using more abstract families of functions to ground one's model often makes deriving implications of a particular type of choice setting easier and the results more general. This is true of the theory of production. It could be said that most of the calculus and algebra of the previous section could have been skipped if students were as comfortable with the abstract modelling methods as the those relying on concrete functional forms.

The easiest applications of the abstract methods are often two- or three-dimensional problems in which there is only a single "degree of freedom" because of the effects of constraints. However, the general approach can be used for any number of variables—although in other cases, matrix methods for derivatives need to be employed and these are rarely used in economic research because the results are often complex, and signs are usually ambiguous—not that ambiguity is never of interest. This text eschews such methods partly for this reason and partly so that more space and class time can be devoted to understanding the implications and foundations of economically relevant decisions that are neglected in other texts.

We'll assume that the production function is from the general family of functions $Q = q(L, K)$ with positive first derivatives, negative second derivatives and positive cross partials. These assumptions assure that the production function of interest is strictly concave, as was true of the concrete functional forms used above. A firm's cost for inputs is again $C = wL + rK$. Given the cost of inputs, we can rewrite the production function as:

$$Q = q(L, (C - wL)/r) \quad (4.35)$$

Differentiating with respect to L to characterize the firm's output maximizing use of labor for a given overall expenditure on inputs (C) can be characterized as:

$$\frac{dQ}{dL} = \frac{dQ}{dL} - \frac{dQ}{dK}\left(-\frac{wL}{r}\right) = 0 \quad (4.36)$$

We can apply the implicit function theorem to characterize any variable in the first-order condition in terms of the other variables in the equation. Each of the partial derivatives of function q includes the same variables as the parent function. Thus, L* can be characterized as:

$$L^* = l(C, w, r) \quad (4.37)$$

Once we know the ideal quantity of labor for a given expenditure (C) on inputs, we can substitute that back into the production function to determine the output produced for a given expenditure and input prices.

$$Q^* = q(L^*, (C - wL^*)/r) \quad (4.38)$$

Equation 4.38 characterizes the output and costs along the firm's production expansion path. Notice that if we subtract either side from the other, we get another zero function.

$$Q^* - q\left(L^*, \frac{C-wL^*}{r}\right) = 0 \equiv H \qquad (4.39)$$

This allows us to apply the implicit function theorem again. In this case, our zero function is of the form $h(Q, C, r, w) = 0$, which we can use to characterize total production costs, C:

$$C = c(Q, w, r) \qquad (4.40)$$

This is the long-run total cost function for the firm of interest. Notice that the cost functions for the multiplicative exponential family of production functions took this form, but they included indicators for technology. To include technology using the abstract approach, a variable such as "T" would be included as an exogenous variable in the production function.

The profit maximizing output, Q*, can be determined in the usual way, as worked out in chapter 3.

$$\Pi = PQ - C$$

$$\frac{d\Pi}{dQ} = P - \frac{dC}{dq} = 0 \; at \; Q^*$$

Hence, $$Q^* = s(P, w, r) \qquad (4.41)$$

A price-taking firm's supply function is a function of the prevailing sales price of its product and its input prices for a given technology of production—as assumed in the abstract model of supply developed in Chapter 3.

The model used in Chapter 3 was not based on ad hoc assumptions, but on the implications of a profit-maximizing firm's production choices in a setting where P is the prevailing price and w and r are the prevailing price of its inputs. Changes in any of these variables will alter the quantity of the good that such a firm will produce for sale.

Input prices affect supply because of their effects on marginal production costs.

VI. A Few General Conclusions

The profit maximizing model has clear implications for the production processes that firms will adopt—namely they will be the least cost method for producing the goods or services that they plan to sell. What the geometry and calculus of production functions demonstrate is that input prices matter—especially when a firm can substitute one input for another in response to changes in input prices. This is not true of every production process, but most have some ability to substitute among their various factors of production.

When substitution is possible, profit maximization implies that the marginal rate of substitution among inputs (which is determined by the technology of production) will equal the relative prices of the inputs used in production. Any deviation from that relationship would imply that profits are not being maximized. Moreover, this will be true as long as profits are maximized, even if the firm owners have not explicitly taken marginal rates of substitution into account. It is a property of profit maximization when inputs are substitutable.

The various families of production functions used to model production provide clear foundations for the models of market supply developed in chapter 3. One does not have to ignore production choices to model supply. An integrated model can be developed fairly easily as demonstrated in this chapter.

The derivations of cost functions undertaken in this chapter make it clear why they include input prices and technology as variables. That inclusion, together with the results from chapter 3, also make it clear why changes in input prices or technology affect market supply, and as demonstrated in the next chapter, why they also affect the prevailing market prices for final goods and services.

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