Part II

I. Introduction: What Has Been Learned from Part I

Part I of the book provides an overview of the main neoclassical models and implications. That model provides a nicely connected schematic of how prices and outputs of the goods and services are determined—or perhaps more accurately, tend to be determined or drawn towards—in well-functioning markets. Prices emerge because terms of trade are part of the normal process of buying and selling goods, whether for money or in exchange for other goods (barter). Such terms of trade, in a sense, determine the prices of all goods and services. What neoclassical theory demonstrates is that in cases where more than one consumer and more than one firm are involved, there are tendencies in these terms of trade that can be deduced if it is assumed that consumers and firms have reasonably stable and consistent aims in life.

To that end, relatively simple models of the aims in life were developed—initially by utilitarians in the late 18th and 19th centuries. These were subsequently adopted by most social scientists that use "rational choice" models to analyze social phenomena. For economists, this form of "rationality" implied that consumers attempt to maximize net benefits or utility. Rational firm owners and managers similarly attempt to maximize profits insofar as production is a means rather than an ultimate end. Firm owners are also utility maximizers in that they use the income realized by their enterprises as a means of increasing their life satisfaction.

As long as the choice settings are well understood (by the persons making decisions and the model-builders characterizing them) most choices in settings of scarcity can be analyzed as applications of constrained optimization. The mathematical implications of constrained optimization, in turn, imply that consumers allocate their resources so that marginal benefits equal marginal costs for each type of good or service purchased. Firm owners similarly allocate their resources to produce the quantities of the things their firms were organized to produce at the levels that are most profitable, which requires minimizing production costs and setting marginal revenues (benefits) equal to marginal costs.

Equilibria in such market networks emerge when prices adjust so that supply equals demand in all of the markets of interest. Neoclassical models provided the logical structure for demonstrating that price vectors exist that can simultaneously "clear" all markets. The logic of single markets can thus be extended to world-wide market networks.

Few neoclassical economists would insist that every market is always in equilibrium (although a few may do so). Rather the models are thought to reveal tendencies that all markets exhibit. Firm owners produce their products at approximately least cost, which requires hiring particular mixes of inputs that are jointly determined by production technology, input prices, and demands for the outputs to be produced. Consumers, likewise, choose combinations of goods that are expected to add most to their lifetime satisfaction (utility), given their wealth, market prices, and their long-term objectives (tastes, preferences, etc.).

In the models developed before WWII, both consumers and firms were usually assumed to have complete (perfect) information about all the factors that generated their choices—even though most thoughtful economists recognized that would rarely if ever be true, except perhaps in markets where the same products had been purchased and sold for many years. In the second half of the twentieth century, more attention was given to the often-unstated informational assumptions made about both firms and consumers during the first half of that century.

Such extensions of the core model lead to more complex models that partially unraveled the neat synthesis that emerged in mid-century. Nonetheless, the neoclassical models continued to provide points of departure for most subsequent research. Both theoretical and empirical economic research generally rest on neoclassical foundations.

II. Overview of Part II

Part II focuses on the subset of extended models that took time and imperfect information into account. These help explain the existence of markets neglected in the core theory and also processes through which new products were introduced and older ones refined. They also explained how markets can improve (or decline) through time. In the core models of neoclassical economics, neither firms nor consumers ever make mistakes. There are no agency problems within firms and no disappointments about the products produced and sold—moreover nothing new is ever produced. Although it was well known that such things happened, the first geometric and mathematical models abstracted from such problems.

These neglected factors were analyzed in what might be called the second-generation models or extended neoclassical models. In many cases, analysis of the effects of imperfect information was undertaken somewhat narrowly and separately, rather than integrated into neoclassical models. Although not thoroughly integrated into the core neoclassical models, the results helped to explain the variation in prices for similar goods that violate some of the conclusions of the perfect competition model. For example, Stigler's (1961) model of the effect of limited knowledge of prevailing prices helped to explain why prices of similar goods usually vary, rather than converge to a single equilibrating number. Similarly, somewhat fuzzier implications were reached about stable patterns of outputs, profits, and of exchange.

The implications of volition in the presence of imperfect information are less sharp, because there are more reasonable ways to make decisions in such choice settings than in the simpler ones analyzed by the neoclassical core. Thus, more types of decisions are made and the patterns that emerge are less uniform.

Part III explores other extensions of the neoclassical models to choice settings that do not directly involve production, exchange, or innovation. It focuses on a subset of such non-economic factors that affect the extent of market networks. Such factors include micro-economic extensions to such fields as law and economics, political economy, and socioeconomics. Besides providing insights into the nature of law, politics, and non-market social interactions, these models also had implications about the extent of trade and the rate at which commerce expands (or contracts) through time.

Although many of those topics are beyond economics, they provide explanations for differences among market equilibria at points in time and through time, shed light on potential sources of disequilibria and adjustments toward new equilibria, and also on why markets differ in their efficiency as institutions for advancing the material interests of those participating in market networks.

Chapter 6: Intertemporal Choice

I. Time, Timing, and Decisions

Most neoclassical models are timeless in the sense that "time" is left out of the model. That is not because time or timing is never important, but because for some purposes leaving time out of a model or analysis does not undermine its ability to help us better understand the puzzle or phenomena being modeled and analyzed. If a consumer decides that he or she will spend one month's wages in a particular way, the fact that the actions associated with that decision do not take place for a week or two does not necessarily influence the optimization process that led to that decision or its consequences for market prices. The period of analysis is simply assumed to be the one that is relevant for the decisions to be made and its associated actions to take place.

However, there are cases in which time matters. Time cannot be ignored when actions are taken today that affect one's possibilities in the future—if one is rational and forward looking. Indeed, the phrase "forward looking" implies that decisionmakers take account of the consequences of present actions on future possibilities. For example, a consumer's decision to borrow money to purchase a house involves decade-long commitments to make mortgage payments in order to receive an even longer potential flow of benefits from housing services. Similarly, a firm may purchase a building and equipment for manufacturing a particular type of good with plans to use both the building and equipment for many years. Servicing loans requires that profits be realized during most of that period.

Given all that, how large a house or factory should one invest in? Clearly the long-term flow of benefits and costs affects both sorts of decisions—as does uncertainty about future income and profits.

Irregularities in future income and profits can lead both consumers and firms to save part of their income to smooth out their consumption. In effect, they transfer some of their purchasing power to the future where it can be spent during periods in which the marginal utility of consumption is higher than in the period in which savings take place. Similarly, many businesses have predictable patterns of profits (net cash flow) that are connected with the seasons and business cycles. They use savings to hold onto their employees during periods of relatively low profits because it increases the firm's productivity by reducing training and recruiting costs. Teams and the knowledge of the routines required for efficient team production benefit from stability of the team members. Examples of seasonal demands include the demand for toys, holiday foods and beverages, and the market for housing (because of school year effects). Firms may, for example, save some of their net income during the most profitable times of the year to pay their employees in less profitable times, because doing so tends to increase annual or decadal profits.

The demand for capital assets, savings, and loans creates a market for a variety of products that would not exist in a timeless world—various types of financial firms emerge to service such desires for intertemporal services. Such firms serve as "intermediaries" between persons desiring to save and those desiring to borrow, with the interest rate or rate of return on investment being the benefit for savers and the cost for borrowers. As a consequence, both loans and savings accounts exist only because of time-dependent interests in saving, long-term investments, and borrowing.

This chapter develops some mathematical methods and models that can be used to characterize intertemporal decision making. Again the focus is on optimization, and again the focus is on circumstances in which buyers and sellers (borrowers and savers) are well-informed about the alternatives before them. It is the effects of time on such decisions and actions that are the main focus of attention. Chapter 7 analyzes the effects of stochastic phenomenon on intertemporal decision making.

For the most part, "optimal" decisionmaking through time rests on the concept of **present discounted value**—the mathematics of which emerges naturally when it is recognized that both borrowing and saving have opportunity costs. Many decisions involve long-term flows of costs and benefits that need to be evaluated by a decision maker or group of decisionmakers. These flows are easiest to compare if one can construct a common "metric" for the purposes of comparison. Present discounted values provide such a means of comparison.

II. Time Discounting and Present Values as an Intertemporal Metric

The simplest way to think about "present discounted value" is to think about the amount in the present (PV) that you would be indifferent to having now rather than some other value (F) in T years.

One way to estimate this, if one thinks in money terms, is to calculate the amount of money that one would have to invest today to have F dollars t years in the future.

- If the interest rate or rate of return is r, one can just apply the compound interest formula. $PV\ (1+r)^t\ =\ F$
- Solving for PV yields $PV\ =\ F_t/(1+r)^t$ which is the basic formula for calculating the present value of some value in the future.
- To make the formula concrete, suppose that F is \$20,000 that t=2 and R=3% or 0.03. In that case, $PV\ =\ (20{,}000)/(1.03)^2\ =\ \$18851.92$
- Notice that PV of future amount F goes down when the interest increases and when the time period increases.
- The PV of \$20,000 in two years at an interest rate of 5% is
  $$PV\ =\ (20{,}000)/(1.05)^2 = \$18{,}140.59$$
- The PV of \$20,000 in ten years at an interest rate of 5% is
  $$PV = (20{,}000)/(1.05)^{10} = \$12{,}278.27$$

If one thinks purely in financial or money terms, one would be indifferent between \$12,278.27 today and \$20,000 in ten years when the interest rate is 5% per annum. This assumes that no inflation occurs (or that F_T is in inflation adjusted terms) and that there is no risk involved about whether the future amount will be paid or not. When one takes account of inflation either everything should be in inflation adjusted (real) terms (including the interest rate, where the real interest rate is the nominal rate of interest less the average annual inflation rate over the period of interest)—or everything should be in nominal (ordinary dollar) terms. When there is a risk that amount F will not be paid or that interest rates will change through time, then one needs to also take account of the risk using the methods developed in Chapter 7.

Let F_t be the value of some asset or income flow "**t**" time periods from the present date. Let **r** be the interest rate per time period over this interval. The present discounted value of Ft is

$$P(F_t) \;=\; F_t/(1+r)^t \qquad (6.1)$$

The present value (here P) of a series of future income flows (which may be positive or negative) is simply the sum of the present values of the individual elements of that series. When calculated over t years when the interest rate is **r** (as a fraction) per period the present value of a series of future values is:

$$P = \sum_{t=1}^{T} F_t\,/(1+r)^t \qquad (6.2)$$

The present value of a series of benefits and/or costs through time is the amount, **P**, that one could deposit in a bank at interest rate **r** and used to replicate the entire stream of benefits or costs, F_1, F_2, F_3, ... F_T. That is to say, you could go to the bank in year 1, withdraw the amount (F_1) for that year, return in year 2, pull out the relevant amount for that year (F_2), and so on ... . When thought of in this way, it should be obvious that the present discounted value of a series of future amounts is simply the sum of the present values of each element of the series—which is equation 6.2.

The present discounted value of any series of values is the sum of the individual present values of each element of the series. This formula always "works" but it is somewhat cumbersome to use as the planning period, t, becomes relatively large.

Another useful formula is one that characterizes the present discounted value of a steady flow of values on and off over the next T years. In cases where a constant value is received through time, e.g. $v = F_1 = F_2 \ldots = F_t \ldots = F_T$, a bit of algebra allows the above present value formula to be reduced to:

$$P \;=\; v\,[(1+r)^T \;-\; 1]\;/\;[r\,(1+r)^T] \qquad (6.3)$$

The derivation is provided in a footnote below.[1]

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[1] This formula can be derived as follows: First multiply $P = \sum_{t=1}^{T} \mathrm{v}/(1+r)^t$ by (1+r) which yields $(1+\mathrm{r})P = \sum_{t=0}^{T-1} v/(1+r)^t$. Subtract the original P from (1+r) P and the

Notice that this constant flow of benefits (or costs) formula **has a limit** as T approaches infinity, namely: $P = v/r$. This is a very convenient formula. There are many long-term investments and regulatory policies that have very long lives that can be thought of as infinitely lived investments as a "first approximation", because the last few billion terms have little effect on the present value of long-term flows of costs or benefits. This formula is simple enough that one can often do the arithmetic in one's head rather than by using a phone, computer or calculator.[2]

The present value formulae are quite flexible and can be used to calculate any single parameter of these equations if all but one of the parameters is known. For example suppose that a bank's opportunity cost rate of return is 5%. Al takes out a mortgage for $300,000. The mortgage is to be repaid in 20 years. What is the lowest annual payment that the bank could possibly demand from Al?

In this case we know P, r and T and need to solve for v. Given $P = v\,[(1+r)^T - 1]\,/\,[r\,(1+r)^T]$, a bit of algebra reveals that

$$v = P[r\,(1+r)^T]/[(1+r)^T - 1] \qquad (6.4)$$

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expression for P from the right-hand variable. Note that all the terms in the difference on the right are the same except for the first and last one, so they cancel out. After doing the subtractions we are left with: $rP = v\,[\,1/(1+r)^0 - 1/(1+r)^T\,)]$.

Recall that $1/(1+r)^0 = 1$. This allows the previous equation to be rewritten as: $rP = v\,[\,1 - 1/(1+r)^T)]$. Putting the lefthand term over a common denominator yields $rP = v\,[\,(1+r)^T - 1]\,/\,[(1+r)^T]$. Dividing both sides by r yields:

$P = v\,[(1+r)^T - 1]\,/\,[r\,(1+r)^T]$ QED

[2] To see this, rewrite equation 6.3 as: $P = v\,[\left(\frac{1}{r}\right) - \left(\frac{1}{r\,(1+r)^T}\right)]$ by separating the numerator and carrying out the division. Notice that as T increases the second term decreases and approaches zero in the limit, leaving only the first term so for very large T, $P = v/r$.

In this case, $v \ = P \ \frac{[r\,(1+r)^T]}{[(1+r)^T \,-\, 1]} = (300000)\,\frac{[(0.05)(1+(0.05))^{20}]}{\left[(1+(0.05))^{20} \,-\, 1\right]} = \$24{,}072.78$. That annual payment, ignoring its administrative costs, would allow the bank to just breakeven. How much above that amount would be required of Al would depend on those costs, the risk of default, and the bank's degree of monopoly power (if any).

Illustrative Application (1)

Suppose that Acme is a price-taking firm and can double the productivity of its present labor force and thus double its present output of 500 units a month by purchasing a machine for $250,000. It sells its output for $5.00 each and the machine is expected to last ten years. If the interest is 10%/year, should Acme purchase the new machine?

Let's put the new revenue in annual terms. The machine will add 500 units per month or 6,000 units a year worth 5x6000=$30,000/year. We can apply the PV formula for a constant stream of benefits to calculate the present value of the new revenue stream:

$$P \ = \frac{(30{,}000)[(1+0.05)^{10} \ - \ 1]}{[(0.05)(1+0.05)^{10}]} = \$231{,}652.04$$

We know the present value of the cost, $250,000, which is greater than present value of the revenue generated. So, Acme should not purchase the new machine if it wants to maximize its profits.

Suppose that the interest rate was 3.5% instead of 5%. In that case, the present discounted value of the additional revenues would have been:

$$P \ = \frac{(30{,}000)[(1+0.035)^{10} \,-\, 1]}{[(0.035)(1+0.035)^{10}]} = \$249{,}498.16$$

In this case, it is close, but the machine is still not worth it. The interest rate would have to be still lower to make the purchase worthwhile. For example, if the interest rate were 3%, the present discounted value of the increase in revenues would be:

$$P \ = \frac{(30{,}000)[(1+0.03)^{10} \ - \ 1]}{[(0.03)(1+0.03)^{10}]} = \$255{,}906.08$$

In which case, the present value of the additional revenue is greater than the present value of the cost of the machine ($250,000), and so purchasing would increase profits. Whether it was the best way to increase output would, of course, depend on whether there were other good options or not.

Illustrative Application (2)

This approach and its associated formulae can also be used for cost-benefit analysis, a systematic method for evaluating the relative merits of public policy. Suppose that a dam can be built that costs $1,000,000 and will produce $50,000/year in electricity for 40 years. Is the dam worth building if the interest rate is 5%/year? We again use the PV formula: $P = v\,[(1+r)^T - 1]\,/\,[r\,(1+r)^T]$. The PV of the future benefits from the dam are

$$P = 50{,}000[((1.05)^{40} - 1)/(.05)(1.05)^{40}] = \$857{,}954.31$$

So, the dam has costs that are greater than its benefits at a 5% interest rate. Were the interest rate lower or were there significant other benefits, then the dam might make sense from a cost-benefit perspective. Note that if the dam would provide electricity forever, then $P = v/r = \$50{,}000/0.05 = \$1{,}000{,}000$. In that case the dam project exactly pay for itself (ignoring any maintenance expenses). But, also note that all the years after year 40 add relatively little to the present discounted value of the future benefits.

III. Intertemporal Choices Using the Present Value Formulae

These sorts of present value calculations can be used to determine the net-benefit maximizing decisions when the annual payoffs and costs are functions of a control variable of interest, such as purchases of a capital good. In many cases, the time-dimension of the choice is not as central as might have been expected. In others, the time dimension is quite important.

Let's revisit Acme's problem and assume that rather than a discrete buy or not choice, Acme can choose from a variety of machines. Acme plans to borrow the money to pay for the machine. The interest is r. Acme's production function is $f(L,K)$ and its initial output is $Q = f(L^0, K^0)$. The cost of additional capital is c(K). Acme's payment on the loan can be calculated from equation 6.4. We'll use an abstract characterization of the fraction of the amount of the loan paid back annually, $b(r,T)$. This allows annual payments to the bank to be written as $b(r,T)c(K)$ where r is the prevailing interest rate and T is the period in which the loan is to be repaid.

The revenue generated (R) varies with the extent of the capital purchased because of its effect on the firm's output. The additional revenue is generated by the additional output, $P[f(L^0,K^0+K)-f(L^0,K^0)]$, where P is the prevailing price of the firm's output, which is assumed to be constant during the planning period. If the life of the project and the loan is T years, the present value of the net benefits or profits from the additional capital:

$$\Pi = \{P[f(L^0,K^0+K)-f(L^0,K^0)]-b(r,T)c(K)][\frac{((1+\mathrm{r})^T-1)}{\mathrm{r}\,(1+\mathrm{r})^T}] \qquad (6.5)$$

Differentiating with respect to K and setting the result equal to zero yields:

$$\Pi_K = [Pf_K - b(r,T)c_K]\left[\frac{[\,((1+\mathrm{r})^T-1)}{\mathrm{r}\,(1+\mathrm{r})^T}\right] = 0 \text{ at } \mathrm{K}^* \qquad (6.6)$$

Multiplying both sides of equation 6.6 by $\left[\frac{\mathrm{r}\,(1+\mathrm{r})^T}{((1+\mathrm{r})^T-1)}\right]$ yields:

$$[Pf_K - b(r,T)c_K] = 0. \qquad (6.7)$$

The first order condition, perhaps surprisingly, is not affected by the term used to calculate present discounted values. The "bottom line" is to purchase additional capital up to the point where its marginal contribution to revenue in each period (its marginal revenue product) equals its marginal cost (the second term).

However, time is not entirely irrelevant because the terms of the loan matter. T and r both affect the marginal cost of capital. And it matters whether marginal revenues exceed marginal costs; unless they do, no additional equipment should be purchased (e.g. if $Pf_K < b(r,T)c_K$). Time is relatively unimportant in this case, even though it is a central feature of the choice setting.

The interest rate matters, but only because this determines the annual cost of the capital good being employed. To maximize the profits from a capital project of this sort, one simply purchases capital so that the annual marginal revenue generated (Pf_K) equals its annual marginal cost ($b(r,T)c_K$).

The same sort of calculation can be undertaken for irregular flows of benefits and costs using the original summation version of the present value formula. In such cases, one would use a variation of equation 6.2 to calculate present values. For example, suppose that the benefits from the capital purchase varied through time—perhaps systematically, perhaps not. In that case, the objective function would be:

$$\Pi = \sum_{t=1}^{T}[Pf_t(K,L) - b(r,T)c(K)]/(1+r)^t \qquad (6.8)$$

The associated first order condition, using subscripts to denote partial derivatives with respect to the variable subscripted (K), is:

$$\Pi_K = \sum_{t=1}^{T} [Pf_{tK} - b(r,T)c_K]/(1+r)^t = 0 \ at \ K^* \qquad (6.9)$$

In this case, present values matter. Acme should set the present discounted value of the marginal revenue (or other marginal benefit) generated by the capital project equal to the present discounted value of the cost of the capital project.

Fluctuations in marginal revenues, for example, could be caused by seasonal effects, changes in excise taxes, or any other unmodelled factor that alters the production function or market prices in a more or less predictable way. The irregularity of the marginal revenue flows is important. As the present discounted value of marginal revenues declines, a smaller capital investment becomes optimal.

IV. Intertemporal Utility Maximization through Borrowing and Saving

The above model can also be applied to consumer choice models based on the net-benefit model developed in Chapter 2. It can also be incorporated into the utility maximizing model, if one believes that lifetime utility function has the form:

$$U = \sum_{t=1}^{T} u_t(X_{1t}, X_{2t}) \,/(1+r)^t. \qquad (6.10)$$

However, one need not adopt a particular concrete functional form for lifetime utility to make use of present values. Intertemporal utility maximization problems generally express the relevant budget constraints in present discounted value terms, with W equal to the present value of future or lifetime income, as with

$$W = \sum_{t=1}^{T} Y_t/(1+r)^t \tag{6.11}$$

and expenditures on goods and services also represented as the present discounted value of future expenditures. An individual's lifetime budget constraint can be characterized by equating the present value of future income (and current wealth), W, with the present value of lifetime expenditures.

$$W = \sum_{t=1}^{T} Y_t/(1+r)^t = \sum_{t=1}^{T}(P_{1t}X_{1t} + P_{2t}X_{2t})/(1+r)^t. \tag{6.12}$$

As with our models of consumer choice and a firm's production decisions, , a good deal about the nature of intertemporal choices can be generated from simple two or three period models of choice. This greatly reduces the mathematical complexity of such models, without much loss of generality.

Two Period Models of Intertemporal Consumption

Suppose that Al's utility function is $U = u(C_1, C_2)$ and her intertemporal budget constraint is $Y_1 + Y_2/(1+r) = C_1 + C_2/(1+r)$, where Y_1 and Y_2 are incomes in period 1 and 2, r is the interest rate or opportunity cost rate of return, and C_1 and C_2 are consumption levels in the two periods. Note that future values are expressed in present value form. This characterization assumes that there is no inflation or that the income and consumption flows and the interest rates are in "real" or inflation adjusted terms.

Both the Lagrangian and substitution methods can be used to characterize Al's optimal consumption expenditure in each period.

Intertemporal Planning with a Concrete Utility Function

We'll begin by assuming that the intertemporal utility function is of the multiplicative exponential variety and that the sum of the exponents is greater than zero but less than one.

Let $U = C_1^a C_2^b$ and let $Y_1 + Y_2/(1+r) - C_1 - C_2/(1+r) = 0$. Form a Lagrange equation and then differentiate with respect to C_1, C_2, and λ.

$$\mathcal{L} = C_1^a C_2^b + \lambda[Y_1 + \frac{Y_2}{1+r} - C_1 - \frac{C_2}{1+r}] \quad (6.13)$$

For the purpose of this model, we'll denote partial derivatives of the Lagrange function with subscripts.

$$\mathcal{L}_{C_1} = aC_1^{a-1}C_2^b - \lambda = 0 \quad (6.14a)$$

$$\mathcal{L}_{C_2} = bC_1^a C_2^{b-1} - \lambda/(1+r) = 0 \quad (6.14b)$$

$$\mathcal{L}_{\lambda} = Y_1 + Y_2/(1+r) - C_1 - C_2/(1+r) = 0 \quad (6.14c)$$

Shift the lambda terms in the first two equations to the right, divide the first equation by the second, and simplify (as usual for this type of function when using the Lagrange method).

$$\frac{aC_1^{a-1}C_2^b}{bC_1^a C_2^{b-1}} = \frac{\lambda}{\lambda/(1+r)}$$

This simplifies to: $\mathrm{aC_2/bC_1 = 1 + r}$

The ratio on the left can be interpreted as the marginal rate of substitution between current and future consumption. The marginal rate of substitution between future and current consumption is sometimes called the *subjective rate of time* discount. The term on the right is the slope of the intertemporal budget constraint. Note that at the utility maximizing levels of C_1 and C_2, the marginal rate of intertemporal substitution is equal to 1 plus the interest rate.

Solve for C_2 as a function of C_1 and then substitute that into the constraint ($\mathcal{L}_\lambda$).

$$C_2 = [b(1+r)C_1/a] \quad (6.15)$$

Substituting yields: $Y_1 + Y_2/(1+r) - C_1 - [b(1+r)C_1/a]/(1+r) = 0$. Shift the C_1 terms to the right (e.g. add the negative of their values to each side) and factor.

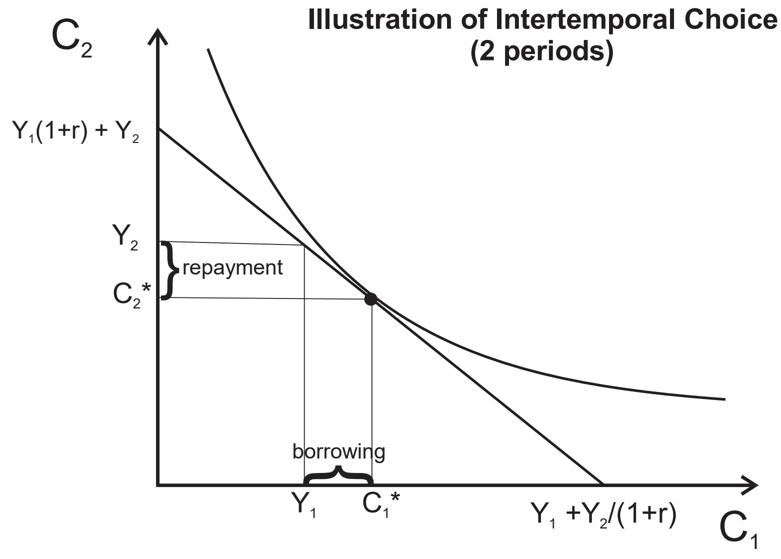
$$Y_1 + Y_2/(1+r) = C_1 + [b(1+r)C_1/a]/(1+r) = C_1\,(1 + b/a) = C_1\,[(a+b)/a]$$

Divide and reverse to find Al's demand curve for present consumption:

$$C_1^* = [a/(a+b)][Y1 + Y2/(1+r)] \quad (6.16)$$

Note that this is analogous to the usual $C_1^* = [a/(a+b)]W/P$ of the non-intertemporal demand functions derived from this family of functions, but here $W = [Y_1 + Y_2/(1+r)]$ and $P = 1 + r$, since we measured wealth as the present discounted value of lifetime income flows, and the relative price of intertemporal consumption was determined by the interest rate. (In effect, the latter implies that we are holding the money prices of current and future consumption constant.)[3]

The geometry of a typical 2-period intertemporal choice is depicted below. The extent of Al's savings is the differences between current income and current consumption, $Y_1 - C_1^*$, which will be negative if he or she borrows against future income to increase his/her current consumption.



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[3] To find C_2^*, substitute C_1^* into equation 6.15. $C2* = [b(1+r)\{[a/(a+b)][Y_1 + Y_2/(1+r)]\}/a]$ which simplifies to $C_2^* = (1+r)[b/(a+b)][Y_1 + Y_2/(1+r)]$. Note the extra (1+r) in the formula for this individual's utility maximizing level of future consumption. Without that term, one would have characterized the present discounted value of future consumption rather than its actual level.

Geometrically, the above model of intertemporal choice looks like an ordinary consumer choice problem except the axes represent present and future consumption.

Note that in the circumstances modeled, the subjective rate of time discount will be set equal to the interest rate plus 1. The indifference curve tangency implies that the slope of the highest indifference curve that can be reached is equal to the slope of the intertemporal budget line. (This is one interpretation of the first steps of solutions to a Lagrangian representation of the choice, as noted above, and it can be derived from the substitution method as well.)

Intertemporal Planning with an Abstract Utility Function

Given an abstract functional form for an individual's utility function, calculus can be used to characterize the effect of changes in interest rates on a person's maximal utility levels and to characterize C_1^* and C_2^*.

Let $U = u(C_1, C_2)$ be an individual's strictly concave utility function and $W = Y_1 + Y_2/(1+r) = \mathrm{C}_1 + \mathrm{C}_2/(1+\mathrm{r})$ be his or her intertemporal budget constraint. Income levels in the two periods are Y_1 and Y_2, and the relevant interest rate is r.

These three variables are assumed to be parameters of the individual's choice problem, which is to say they are assumed to be exogenously determined as they would be if they were determined by international markets. The individual's choice is assumed to be over the timing of consumption. (In other cases, decisions that affect future income may also be possible, as with one's investment in a college education.)

Since there are just two control variables and one constraint, we can solve the constraint for one of the two control variables in terms of the other and substitute it into the individual's utility function to simplify the calculus. For example, we can solve for C_2 as:

$$C_2 = (1+r)Y_1 + Y_2 - (1+r)C_1. \qquad (6.17)$$

Substituting this into the utility function yields:

$$U = u(C_1, (1+r)Y_1 + Y_2 - (1+r)C_1). \qquad (6.18)$$

Differentiating with respect to C_1 yields:

$$dU/dC_1 - (1+r)dU/dC_2 = 0 \ at \ C_1^* \tag{6.19}$$

Note that the first term, dU/dC_1, is the marginal benefit from current consumption and the second is its subjective marginal cost,$(1+r)du/dC_2$. A forward-looking consumer consumes in the current period (today) at the level that equates his or her marginal benefits today with its marginal cost, here measured in terms of reduced utility from future consumption.

A bit of algebra also allows one to use the first order condition to characterize the tangency condition of an indifference curve diagram and the intertemporal budget constraint.

$$\frac{\left(\frac{dU}{dC_1}\right)}{\left(\frac{dU}{dC_2}\right)} = (1+r) \tag{6.20}$$

The implicit function theorem allows consumption in period 1 to be characterized as a function of the parameters of the individual's intertemporal choice setting:

$$C_1^* = c(Y_1, Y_2, r) \tag{6.21}$$

with $C_2^* = (1+r)Y_1 + Y_2 - (1+r)C_1^*$ (6.22)

An individual's intertemporal pattern of consumption is a function of his or her present and future income and the interest rate.

The comparative statics of the individual's choice can be characterized using the implicit function differentiation rule. For example, the effect of an anticipated increase in the interest rate on current consumption is:

$$dC_1^*/dr = \frac{dH/dr}{-dH/dC_1} =$$

$$\frac{[(Y_1-C_1)(d^2U/dC_1dC_2) - dU/dC_2 - (1+r)(Y_1-C_1)dU^2/dC_2^2\,]}{-dHdC1} \tag{6.23}$$

where $dH/dC_1 = d^2U/dC_1^2 - 2(1+r)d^2U/dC_1dC_2 + (1+r)^2d^2U/dC_2^2] < 0.$

The sign is determined by the numerator, because the denominator is positive, given the strict concavity assumed. Note that the numerator can be greater or less than zero depending on whether the individual is a borrower or a saver in period 1.

If the person modelled saves, then $Y - C_1 > 0$ and the numerator is greater than zero if the first and last term dominate the middle term. In that case, the overall effect of an increase in the interest rate on savers is to increase current consumption. Intuitively, this is because an increase in interest rates increases lifetime income for savers, and that increase, together with an increase in the returns from saving induces them to "spread the wealth" between the current and future consumption. On the other hand, if the individual is a borrower in period 1, the effect on consumption in period 1 is ambiguous. In that case, $Y - C_1 < 0$, and all three terms in the numerator are negative. Borrowers thus reduce their current spending because the marginal opportunity cost has increased.

The effect of an increase in current income on current consumption can be developed in a similar way:

$$dC_1^*/dY_1 = \frac{(dH/dY_1)}{-dH/dC_1} =$$

$$(1+r)\frac{\frac{d^2U}{dC_1dC_2}-(1+r)(C_1)\frac{d^2U}{dC_2^2}}{-\frac{dH}{dC_1}} > 0 \qquad (6.24)$$

The numerator in this case is positive and the denominator (which is the same as in the previous derivation) is positive. Similar results would hold for an anticipated increase in future income. An increase in income, thus, unambiguously tends to increase current and future consumption.

These quite general findings imply that interest rates and expectations about future income are both important determinants of current consumption. Time has consequences.

Extensions

Both the explicit functional form model and the abstract functional form model can be extended to characterize multiple periods. A multi-period utility function can be generated for the multiplicative form or abstract forms for utility function used above, by adding additional "C" terms for periods, 3, 4…and T with associated exponents in the multiplicative exponential case. The budget constraint in either case would set the present value of lifetime expenditures on the goods under consideration to the present value of income flows during the same planning period—which could be a lifetime.

Another possible extension is to consider the possibility of continuous flows of utility and income rather than discrete flows.

V. The Market for Savings and Loans in a Setting without Risk

The intertemporal consumption models imply that there may be gains to trade between persons who wish to borrow in the present because their income is less than their desired consumption ($Y_1 < C_1^*$) and those that wish to save because their income is greater than their desired consumption ($Y_1 > C_1^*$). Borrowing is possible when future income is sufficient to pay back the loan and interest rates are sufficient to induce sufficient saving to provide the basis for such loans.

In informal financial markets, such persons might simply meet up with each other (as still occasionally happens) and the person seeking a loan (the borrower) would receive one from the person willing to make a loan (the saver). In the risk-free environment assumed to this point in the book, such agreements would be relatively easy to consummate, but they might still take significant time to work out if the borrower wants a larger loan than any single saver is willing to make—as would likely be the case for loans to purchase a house or condominium.

As more formal financial markets emerge, this matching process is undertaken by various "middleman" firms (referred to as financial intermediaries), who would pay savers (r^s) for the temporary use of their savings and charge a somewhat more than that amount (r^b) for individuals who sought to borrow some money. Banks are one example of such firms—but there are many others that vary partly because of differences in the riskiness of the returns for savers and the riskiness of those receiving loans. We'll ignore the effects of risk for now; analyzing those effects are taken up in chapter 7.

For now, we'll continue to assume that all is known, and so there is no risk for the intermediaries nor for those making loans to them. The borrowers are all trustworthy and have sufficient funds to repay the loans in the future, and the intermediaries are honest or sufficiently fearful of penalties for fraud to behave as if they were honest.

The difference between the amounts paid to those depositing funds in the bank (those loaning the bank their money) and those borrowing the money from the bank reflects the bank's cost of doing business as well as the value added (reduced transaction costs) by their services.

In a world of complete information and certainty, banks would resemble Marshallian firms. They would tend to use very similar technologies and inputs. In such cases, competition would induce what might be called their middleman fees ($r^b - r^s$) to converge toward ones that are equal to the average cost of collecting deposits, assuring their safety, assessing the trustworthiness and future income of the persons taking out loans, and keeping accurate records plus the "normal" return on capital (e.g. a financial intermediary's investments in vaults, buildings, computers, and so forth).

Competition does not, however, reduce the markup to zero, because of the various costs that financial intermediaries bear to provide their services. The interest rates for borrowers are necessarily higher than that paid to savers ($r^b > r^s$), because the services provided by banks and other similar organizations are costly to produce even in a risk-free choice setting. In cases in which a bank has some monopoly power, as might be the case in towns with only a few banks, there will be a markup beyond that required to cover its costs and provide an "ordinary" rate of return on the capital used to provide the intermediary (banking) services.

Except for the name for the price paid for inputs and outputs (interest paid to depositors and interest charged borrowers), the market for intermediation services is a straight-forward application of the theory of the firm worked out in chapter 3, with some intertemporal aspects, and the price theory worked out in chapter 5. In this case, the prices (r^b and r^s) are determined by the demand for saving and demand for loans, and their difference by the cost of producing intermediation services.

The steady state size of a bank (supply of loans, Q) attempts to maximize the present value of profits from providing those services over its planning horizon T:

$$\Pi = \sum_{t=1}^{T}[(r^b - r^s)Q_t - c(Q,w,r)]/(1+r)^t \qquad (6.25)$$

The associated first order condition is:

$$\Pi_Q = \sum_{t=1}^{T} \left[(r^b - r^s) - c_Q\right]/(1+r)^t = 0 \; at \; Q^* \qquad (6.26)$$

In a steady state $[(r^b - r^s) - c_Q]$ is a constant and can be factored out to yield:

$$[(r^b - r^s) - c_Q]\left[\sum_{t=1}^{T} \frac{1}{(1+r)^t}\right] = 0$$

Which reduces to:

$$\left[(r^b - r^s) - c_Q\right] = 0 \; at \; Q^* \qquad (6.27)$$

The bank will have a portfolio of loans that equates its marginal revenue from loans to its marginal costs of providing and servicing those loans and their associated deposits.

Note that in this case, the result is more or less the same as would have been the case for firms in the time-less models worked out in part 1.[4]

The Demand and Supply of Credit

The demand and supply of credit in a setting where both suppliers and demanders are price takers requires two markets to clear, the market for loans (populated by borrowers) and the market for savings. The demand side of the loan market is populated by borrowers. The supply side of this market is populated by savers. Banks and other financial intermediaries are input purchasers in the savings market and final producers of the loan market.

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[4] Note that this characterization is not very different from those used in basic macroeconomic models, although in those cases general equilibrium effects on income are also taken into account—usually by assuming specific functional forms for the linkages between interest rates, savings, borrowing, and investment income. Here, we undertake a partial equilibrium analysis, but with a generalized model of intermediation (banking and the like).

The results of the previous two sections imply that the market supply of savings is an increasing function of the interest rate (r^s), whereas the demand for that input to the intermediating firms is a decreasing function of that rate. Conversely, the supply of loans from the intermediary is an increasing function of the borrowing rate, whereas its demand is a decreasing function of the interest rate (r^b). The market clearing interest rates in those two markets determine the equilibrium difference between the saving interest rate and the borrowing interest rate. Intermediaries adapt to that difference and operate at the scale implied by equation 6.27.

VI. Conclusions

Many of the conclusions worked out in the timeless models of the firm and consumer in Part I do not change very much by taking account the pattern of future benefits and costs generated by a current decision. This is particularly true of choices in which the future flows of benefits and costs are stable. It is less true of cases in which the pattern of future costs and benefits is irregular. The basic logic of the models of chapter one can often be maintained by replacing the term benefits with the present discounted value of benefits and the term cost with the present discounted value of costs. In such cases, taking account of intertemporal aspects of the choices being made simply deepen the analysis without changing many of the previous conclusions. In equilibrium the present value of marginal costs equals the present value of marginal benefits for the decision makers of interest—rather than marginal costs equaling marginal benefits, per se.

Other implications are new—they are ones that cannot be developed in a timeless model. Savings, borrowing, and financial intermediation services do not exist in a timeless world. Although such markets are neglected in a timeless world, once recognized the prices that emerge in such markets (interest rates) can be modelled in much the same way that they were in "timeless" markets as opposed to markets where time-shifting is the service being provided. There are, for example, monopoly and competitive forms of markets for intermediation, with different implications for the prices that emerge.

It is interesting to note that the demand for "income smoothing" can account for the existence of markets for intermediation—even in settings where there is no risk and no uncertainties. The demand for income smoothing is simply an implication of diminishing marginal utility. One can increase overall utility by shifting income in periods where the marginal utility of the last dollar of consumption is low (high income periods) to periods in which it is higher (lower income periods). That relationship, together with well-known patterns of income is sufficient to generate markets for intermediation services in settings where there is no fraud and contracts are perfectly enforced.

That demand is increased by risk and uncertainty, but not caused by it, as developed in the next chapter. Indeed, the demand for insurance is simply another instance of the demand for income smoothing.

References

Boardman, A. E., Greenberg, D. H., Vining, A. R., & Weimer, D. L. (2018). *Cost-benefit analysis: Concepts and practice* (5th ed.). Cambridge University Press.

Conard, J. W. (2023). *Introduction to the Theory of Interest*. University of California Press.

Ferejohn, J., & Page, T. (1978). On the foundations of intertemporal choice. *American Journal of Agricultural Economics*, *60*(2), 269-275.

Fisher, I. (1930). *The theory of interest*. Macmillan.

Friedman, M. (1957). *A theory of the consumption function*. Princeton University Press.

Homer, S., & Sylla, R. E. (1996). *A history of interest rates*. Rutgers University Press.

Laibson, D. (1997). Golden eggs and hyperbolic discounting. *Quarterly Journal of Economics*, *112*(2), 443–477.

Loewenstein, G., & Thaler, R. H. (1989). Anomalies: intertemporal choice. *Journal of Economic Perspectives*, *3*(4), 181-193.

Modigliani, F., & Brumberg, R. (1954). Utility analysis and the consumption function: An interpretation of cross-section data. In K. Kurihara (Ed.), *Post-Keynesian economics* (pp. 388–436). Rutgers University Press.

Rutterford, J. (2004). From dividend yield to discounted cash flow: a history of UK and US equity valuation techniques. *Accounting, Business & Financial History*, *14*(2), 115-149.

Samuelson, P. A. (1994). Two Classics: Böhm-Bawerk's positive theory and Fisher's rate of interest through modern prisms. *Journal of the History of Economic Thought*, *16*(2), 202-228.

Shackle, G. L. S. (2010). *Decision order and time in human affairs*. Cambridge University Press.