# Answers for Problems Assigned Week 3

**1.** Use the Lagrangian and Substitution Methods to characterize the utility maximizing level of goods G and H in the case where

**A. Substitution Method**

i. maximize U = HG where W = pG + H

- substituting yields: $U = (W\text{-}pG)G = GW - pG^2$
- differentiating with respect to G yields: $W - 2pG = 0$ (at G*)
- solving for G* yields $G^* = W/2p$

ii. $U = H(G\text{-}2)^2$ where $10 = G + 2H$

a. Note that G = 10 - 2H

- substituting yields: $U = H(8\text{-}2H)^2$
- differentiating with respect to H yields:
- $U_H = (8\text{-}2H)^2 + 2H(8\text{-}2H)(\text{-}2) = 0$ (use the composite function and multiplicative rules.)
- $U_H = 4H^2 - 32H + 64 - 32H + 8H^2 = 0$ (at H*)
- $U_H = 12H^2 - 64H + 64 = 3H^2 - 16H - 16 = 0$

b. The quadratic formula implies

- $H = [16 +/- (256 - 192)^{1/2}] / 6$
- so:
- $H^* = 16/6 +/- 8/6 = \{ 4/3, 4 \}$
- and
- $G^* = 10 - 2H^*$ or $\{ 22/3, 2 \}$

iii. U = HG where $H^2 + G^2 = 10$

a. Note that constrain implies $H = +/- (10\text{-}G^2)^{1/2}$

- so $U = +/- G (10\text{-}G^2)^{1/2}$

b. Differentiating with respect to G yields

- $(10\text{-}G^2)^{1/2} + (1/2)G(10\text{-}G^2)^{-1/2}(\text{-}2G) = 0$ at G*

c. Multiplying by $(10\text{-}G^2)^{1/2}$ to simplify yields:

- $(10\text{-}G^2) - G^2 = 0$ or $2G^2 = 10$
- so $G^* = +/- (5)^{1/2}$ (plus or minus the square root of 5)

d. Substituting G* into the equation characterizing H as a function of G yields;

- $H^* = +/- (10\text{-}G^{*2})^{1/2} = +/- (5)^{1/2}$

**B. Lagrangian Method**

i. U = HG where W = pG + H

- Form a Lagrangian: $L = HG - \lambda(W - pG - H)$
- differentiating yields: $H - \lambda p = 0$
- $G - \lambda = 0$
- and: $W - pG - H = 0$
- Now solve for H, G, and $\lambda$.
- dividing the first two yields: $H/G = p$ or $H = Gp$
- which implies that $W - pG - pG = 0$ or $G^* = W/2p$
- and $H = W/2$

ii. Given: $U = H(G\text{-}2)^2$ where $10 = G + 2H$

- Form a Lagrangian: $L = H(G\text{-}2)^2 - \lambda(10 - G - 2H)$
- Differentiate with respect to H and G (the control variables) and $\lambda$ which yields three equations:
- $L_H = (G\text{-}2)^2 - 2\lambda = 0$
- $L_G = 2GH - 4H - \lambda = 0$
- $L_\lambda = (10 - G + 2H) = 0$

a. Next, solve this system of simultaneous equations for H and G.

- Dividing a/b and solving for G yields $G = 4H + 2$
- Substituting into the constraint for G yields

  $10 = 4H^* - 2 + 2H^*$ or $H^* = 8/6 = 4/3$
- and since $G^* = 10 - 2H^*$ (from the constraint) $G^* = 22/3$
- (You have to be on the constraint under Lagrange. If you had just substituted back into G = 4H + 2 you would be describing a point on the consumption expansion path. Both methods should yield the same result, because the solution is where the consumption expansion path crosses the budget line.)

iii. Given U = HG where $H^2 + G^2 = 10$

- Form a Lagrangian: $L = HG + \lambda( 10 - H^2 - G^2 )$
- Differentiate with respect to control variables H and G and the Lagrangian multiplier $\lambda$
- $L_H = G - 2H\lambda = 0$
- $L_G = H - 2G\lambda = 0$
- $L_\lambda = 10 - H^2 - G^2 = 0$

a. Solve this system of simultaneous equation ( c, d, e) for G and H. (You can also solve for the Lagrangian multiplier if you wish, but this is optional.)

- Dividing c and d, then cross multiplying yields $H^2 = G^2$
- or $G = +/- H$
- Substituting into the constraint for G yields: $H^2 + H^2 = 10$
- or $H^* = +/- (5)^{1/2}$

b. That is to say, H* equals plus or minus the square root of 5.

c. Since from "c," $G^* = +/- H^*$ $G^*$ also equals $+/- (5)^{1/2}$

**2.** Use the Kuhn-Tucker Technique to characterize the utility maximizing level of goods W and L when:

i. $U = L^{.5} + 1/(1+W)$ and $24 \geq W + L$ with $W, L \geq 0$

- Note that U is strictly concave and the feasable set is convex. The second derivatives of U are both negative, and the cross partial is greater than or equal to zero.

a. Form the KT function in a manner similar to that of the Lagrangian:

- $K = L^{.5} + (1+W)^{-1} + \lambda ( 24 - W - L )$

b. Differentiating with respect to the control variables W and L (work and leisure) and with respect to $\lambda$ yields the following KT first order conditions:

c. $K_L = .5 L^{-.5} - \lambda <= 0$ with $L \geq 0$ and $L ( .5 L^{-.5} - \lambda) = 0$

d. $K_W = - (1+W)^{-2} - \lambda < = 0$ with $W \geq 0$ and $W ( - (1+W)^{-2} ) = 0$

e. $K_\lambda = 24 - W - L \geq 0$ with $\lambda \geq 0$ and $\lambda (24 - W - L) = 0$

- Before working on the KT conditions, take a moment to think about the objective function. Note that utility rises with L and *falls with W*. W is a bad. This person will consume as little as possible of the bad, W, and as much as possible of the good, L. The constraints determine what is possible.

f. Now consider whether W = 0 and L = 24 satisfies the KT conditions.

- Equation e, the constraint, is clearly satisfied--as an equality.
- Equation d, the f. o. c. for W is also clearly satisfied. The negative of a squared term is always negative, unless it equals zero. With W= 0, we are still OK as long as $\lambda$ is positive or a small negative number we are all right here. In that case, W=0 satisfies the non-negativity constraint and the product constraint.
- Equation c is satisfied only if $\lambda$ takes the value $.5 (24)^{-.5}$ in which case the $K_L = 0$. If L= 24, then L is clearly greater than zero, and our assumption about $\lambda$ assures that $K_L = .5 L^{-.5} - \lambda = 0$. Given this value for $\lambda$, the product constraint is satisfied.
- We have now shown that the combination W = 0, L = 24, $\lambda = .5 L^{-.5}$ satisfy the KT first order conditions.
- QED
- Given U quasi concave, and the convex opportunity set, the combination W = 0, L = 24 , $\lambda = .5 (24)^{-.5}$ ) satisfies the KT first order conditions for a maximum
- (The Arrow Enthoven sufficiency theorem implies that given a convex feasible and quasi-concave objective function, the KT first order conditions are sufficient to characterize the constrained optimal values of the control variables.)

ii. Interpret your result. What kind of "good" is W ?

- As noted above W is a bad, perhaps unpleasant and unpaid work in a setting where leisure is an option. Consequently, this individual chooses a corner solution with L = 24 and W = 0.